DIRECTORATE OF DISTANCE EDUCATION

UNIVERSITY OF NORTH BENGAL

MASTER OF SCIENCE -MATHEMATICS

SEMESTER -II

REAL ANALYSIS

DEMATH2-CORE 1

BLOCK-2

UNIVERSITY OF NORTH BENGAL

Postal Address:

The Registrar,

University of North Bengal,

Raja Rammohunpur,

P.O.-N.B.U., Dist-Darjeeling,

West Bengal, Pin-734013,

India.

Phone: (O) +91 0353-2776331/2699008

Fax: (0353) 2776313, 2699001

Email: regnbu@sancharnet.in ; regnbu@nbu.ac.in

Wesbsite: www.nbu.ac.in

First Published in 2019

ENLIGHTENMENT TO PERFECTION

FOREWORD

The Self Learning Material (SLM) is written with the aim of providing simple and organized study content to all the learners. The SLMs are prepared on the framework of being mutually cohesive, internally consistent and structured as per the university's syllabi. It is a humble attempt to give glimpses of the various approaches and dimensions to the topic of study and to kindle the learner's interest to the subject

We have tried to put together information from various sources into this book that has been written in an engaging style with interesting and relevant examples. It introduces you to the insights of subject concepts and theories and presents them in a way that is easy to understand and comprehend.

We always believe in continuous improvement and would periodically update the content in the very interest of the learners. It may be added that despite enormous efforts and coordination, there is every possibility for some omission or inadequacy in few areas or topics, which would definitely be rectified in future.

We hope you enjoy learning from this book and the experience truly enrich your learning and help you to advance in your career and future endeavours.

REAL ANALYSIS

BLOCK-1

BLOCK-2

BLOCK-1 REAL ANALYSIS

Introduction to the block

Unit-1 Introduction to Real Numbers: After studying this unit, you will learn about Real Numbers, Understand what are Numbers, how to work with Real Numbers.

Unit-2 Basic Set Theory: After studying this unit, you will learn about what are sets, working with set theory and Construction of sets

Unit-3 Extended Real Numbers: In this unit you will study need of real numbers with different perspectives. Unit also covers measure and integration and real projective plane topics.

Unit-4 Algebraic Operations: In this unit you will learn meaning of algebra, Signed numbers, Algebraic equations & basic approach to solving algebraic word problems.

Unit-5 Sequence and Convergence: In this unit you will study meaning of sequence and convergence, limits, Cauchy sequence and related theorems

Unit-6 Lebesgue Measure: This unit will introduce you to Lebesgue measure and its properties, null sets and Heine-Borel theorem.

Unit-7 Lebesgue Outer Measure: In this unit you will study Lebesgue outer measure & measurability.

UNIT - 1: INTRODUCTION TO REAL NUMBERS

STRUCTURE

1.0 OBJECTIVES

After studying this unit, you should be able to:

- learn about Real Numbers
- Understand what is Numbers
- Work with Real Numbers
- Construction with Rational Numbers

1.1 INTRODUCTION

In mathematics, a **real number** is a value of a continuous quantity that can represent a distance along a line. The adjective *real* in this context was introduced in the 17th century by René Descartes, who distinguished between real and imaginary roots of polynomials. The real numbers include all the rational numbers, such as the integer −5 and the fraction 4/3, and all the irrational numbers, such as $\sqrt{2}$ (1.41421356..., the square root of 2, an irrational algebraic number). Included within the irrationals are the transcendental numbers, such as π (3.14159265...). In addition to measuring distance, real numbers can be used to measure quantities such as time, mass, energy, velocity, and many more.

The set of all real numbers is uncountable; that is: while both the set of all natural numbers and the set of all real numbers are infinite sets, there can be no one-to-one function from the real numbers to the natural numbers: the cardinality of the set of all real numbers (denoted and called cardinality of the continuum) is strictly greater than the cardinality of the set of all natural numbers (denoted 'aleph-naught'). The statement that there is no subset of the real’s with cardinality strictly greater than and strictly smaller than is known as the continuum hypothesis (CH). It is known to be neither provable nor refutable using the axioms of Zermelo–Fraenkel set theory including the axiom of choice (ZFC), the standard foundation of modern mathematics, in the sense that some models of ZFC satisfy CH, while others violate it.

1.2 DEFINITION REAL NUMBERS

The real number system R can be defined axiomatically up to an isomorphism, which is described hereafter. There are also many ways to construct "the" real number system, for example, starting from natural numbers, then defining rational numbers algebraically, and finally defining real numbers as equivalence classes of their sequences or as Dedekind cuts, which are certain subsets of rational numbers. Another possibility is to start from some rigorous axiomatization of Euclidean geometry (Hilbert, Tarski, etc.) and then define the real number system geometrically. All these constructions of the real numbers have been shown to be equivalent, that is the resulting number systems are isomorphic

1.2.1 Axiomatic approach

Let **R** denote the set of all real numbers. Then:

- The set **R** is a field, meaning that addition and multiplication are defined and have the usual properties.
- The field **R** is ordered, meaning that there is a total order x and y such that, for all real numbers x, y and z:
  - if $x \geq y$ then $x + z \geq y + z$;
  - If $x \geq 0$ and $y \geq 0$ then $xy \geq 0$.
- The order is Dedekind-complete; that is: every non-empty subset S of **R** with an upper bound in **R** has a least upper bound (also called supremum) in **R**.

The last property is what differentiates the real's from the rationals (and from other, more exotic ordered fields). For example, the set of rationals with square less than 2 has a rational upper bound (e.g., 1.5) but no rational least upper bound, because the square root of 2 is not rational.

These properties imply Archimedean property (which is not implied by other definitions of completeness). That is, the set of integers is not upper-bounded in the real's. In fact, if this were false, then the integers would have a least upper bound N; then, $N – 1$ would not be an upper bound, and there would be an integer n such that $n > N – 1$, and thus $n + 1 > N$, which is a contradiction with the upper-bound property of N.

The real numbers are uniquely specified by the above properties. More precisely, given any two Dedekind-complete ordered fields $\mathbf{R}_1$ and $\mathbf{R}_2$, there exists a unique field isomorphism from $\mathbf{R}_1$ to $\mathbf{R}_2$, allowing us to think of them as essentially the same mathematical object.

For another axiomatization of $\mathbb{R}$, see Tarski's axiomatization of the real's.

1.3 PROPERTIES

1.3.1 Basic Properties

- Any non-zero real number is either negative or positive.
- The sum and the product of two non-negative real numbers is again a non-negative real number, i.e., they are closed under these operations, and form a positive cone, thereby giving rise to a linear order of the real numbers along a number line.
- The real numbers make up an infinite set of numbers that cannot be injectively mapped to the infinite set of natural numbers, i.e., there are uncountable infinitely many real numbers, whereas the natural numbers are called countable infinite. This establishes that in some sense, there are more real numbers than there are elements in any countable set.
- There is a hierarchy of countable infinite subsets of the real numbers, e.g., the integers, the rationals, the algebraic numbers and the computable numbers, each set being a proper subset of the next in the sequence. The complements of all these sets (irrational, transcendental, and non-computable real numbers) with respect to the reals are all unaccountably infinite sets.
- Real numbers can be used to express measurements of continuous quantities. They may be expressed by decimal representations, most of them having an infinite sequence of digits to the right of the decimal point; these are often represented like 324.823122147..., where the ellipsis (three dots) indicates that there would still be more digits to come. This hints to the fact that we can precisely denote only a few, selected real numbers with finitely many symbols.

More formally, the real numbers have the two basic properties of being an ordered field, and having the least upper bound property. The first says that real numbers comprise a field, with addition and multiplication as well as division by non-zero numbers, which can be totally ordered on a number line in a way compatible with addition and multiplication. The second says that, if a non-empty set of real numbers has an upper bound, then it has a real least upper bound. The second condition distinguishes the real numbers from the rational numbers: for example, the set of rational numbers whose square is less than 2 is a set with an upper bound (e.g. 1.5) but no (rational) least upper bound: hence the rational numbers do not satisfy the least upper bound property.

1.3.2 Completeness

A main reason for using real numbers is that the real's contain all limits. More precisely, a sequence of real numbers has a limit, which is a real number, if (and only if) its elements eventually come and remain arbitrarily close to each other. This is formally defined in the following, and means that the real's are complete (in the sense of metric spaces or uniform spaces, which is a different sense than the Dedekind completeness of the order in the previous section). :

A sequence (x_n) of real numbers is called a Cauchy sequence if for any $\varepsilon > 0$ there exists an integer N (possibly depending on ε) such that the distance $|x_n - x_m|$ is less than ε for all n and m that are both greater than N. This definition, originally provided by Cauchy, formalizes the fact that the x_n eventually come and remain arbitrarily close to each other.

A sequence (x_n) converges to the limit x if its elements eventually come and remain arbitrarily close to x, that is, if for any $\varepsilon > 0$ there exists an integer N (possibly depending on ε) such that the distance $|x_n - x|$ is less than ε for n greater than N.

Every convergent sequence is a Cauchy sequence, and the converse is true for real numbers, and this means that the topological space of the real numbers is complete.

The set of rational numbers is not complete. For example, the sequence (1; 1.4; 1.41; 1.414; 1.4142; 1.41421; ...), where each term adds a digit of the decimal expansion of the positive square root of 2, is Cauchy but it does not converge to a rational number (in the real numbers, in contrast, it converges to the positive square root of 2).

The completeness property of the real's is the basis on which calculus, and, more generally mathematical analysis are built. In particular, the test that a sequence is a Cauchy sequence allows proving that a sequence has a limit, without computing it, and even without knowing it.

For example, the standard series of the exponential function

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Converges to a real number for every x, because the sums

$$\sum_{n=N}^{M} \frac{x^n}{n!}$$

Can be made arbitrarily small (independently of M) by
choosing N sufficiently large. This proves that the sequence is Cauchy, and thus converges, showing that is e^x well defined for every x.

1.3.3 The Complete Ordered Field

The real numbers are often described as "the complete ordered field", a phrase that can be interpreted in several ways.

First, an order can be lattice-complete. It is easy to see that no ordered field can be lattice-complete, because it can have no largest element (given any element z, z + 1 is larger), so this is not the sense that is meant.

Additionally, an order can be Dedekind-complete, as defined in the section **Axioms**. The uniqueness result at the end of that section justifies using the word "the" in the phrase "complete ordered field" when this is the sense of "complete" that is meant. This sense of completeness is most closely related to the construction of the real's from Dedekind cuts, since that construction starts from an ordered field (the rationals) and then forms the Dedekind-completion of it in a standard way.

These two notions of completeness ignore the field structure. However, an ordered group (in this case, the additive group of the field) defines a uniform structure, and uniform structures have a notion of completeness (topology); the description in the previous section **Completeness** is a special case. (We refer to the notion of completeness in uniform spaces rather than the related and better known notion for metric spaces, since the definition of metric space relies on already having a characterization of the real numbers.) It is not true that **R** is the only uniformly complete ordered field, but it is the only uniformly complete Archimedean field, and indeed one often hears the phrase "complete Archimedean field" instead of "complete ordered field". Every uniformly complete Archimedean field must also be Dedekind-complete (and vice versa), justifying using "the" in the phrase "the complete Archimedean field". This sense of completeness is most closely related to the construction of the real's from Cauchy sequences (the construction carried out in full in this article), since it starts with an Archimedean field (the rationals) and forms the uniform completion of it in a standard way.

But the original use of the phrase "complete Archimedean field" was by David Hilbert, who meant still something else by it. He meant that the real numbers form the largest Archimedean field in the sense that every other Archimedean field is a subfield of **R**. Thus **R** is "complete" in the sense that nothing further can be added to it without making it no longer an Archimedean field. This sense of completeness is most closely related to the construction of the real's from surreal numbers, since that construction starts with a proper class that contains every ordered field (the surreal) and then selects from it the largest Archimedean subfield.

1.3.4 Advanced Properties

The reals are uncountable; that is: there are strictly more real numbers than natural numbers, even though both sets are infinite. In fact, the cardinality of the real's equals that of the set of subsets (i.e. the power set) of the natural numbers, and Cantor's diagonal argument states that the latter set's cardinality is strictly greater than the cardinality of **N**. Since the set of algebraic numbers is countable, almost all real numbers are transcendental. The non-existence of a subset of the real's with cardinality strictly between that of the integers and the real's is known as the continuum hypothesis. The continuum hypothesis can neither be proved nor be disproved; it is independent from the axioms of set theory.

As a topological space, the real numbers are separable. This is because the set of rationals, which is countable, is dense in the real numbers. The irrational numbers are also dense in the real numbers; however they are uncountable and have the same cardinality as the real's.

The real numbers form a metric space: the distance between x and y is defined as the absolute value |x − y|. By virtue of being a totally ordered set, they also carry an order topology; the topology arising from the metric and the one arising from the order are identical, but yield different presentations for the topology in the order topology as ordered intervals, in the metric topology as epsilon-balls. The Dedekind cuts construction uses the order topology presentation, while the Cauchy sequences construction uses the metric topology presentation. The real's are a contractible (hence connected and simply connected), separable and complete metric space of Hausdorff dimension 1. The real numbers are locally compact but not compact. There are various properties that uniquely specify them; for instance, all unbounded, connected, and separable order topologies are necessarily homeomorphic to the real's.

Every nonnegative real number has a square root in **R**, although no negative number does. This shows that the order on **R** is determined by its algebraic structure. Also, every polynomial of odd degree admits at least one real root: these two properties make **R** the premier example of a real closed field. Proving this is the first half of one proof of the fundamental theorem of algebra.

The real's carry a canonical measure, the Lebesgue measure, which is the Haar measure on their structure as a topological group normalized such that the unit interval [0;1] has measure 1. There exist sets of real numbers that are not Lebesgue measurable, e.g. Vitali sets.

The supremum axiom of the real's refers to subsets of the real's and is therefore a second-order logical statement. It is not possible to characterize the real's with first-order logic alone: the Löwenheim–Skolem theorem implies that there exists a countable dense subset of the real numbers satisfying exactly the same sentences in first-order logic as the real numbers themselves. The set of hyper real numbers satisfies the same first order sentences as **R**. Ordered fields that satisfy the same first-order sentences as **R** are called nonstandard models of **R**. This is what makes nonstandard analysis work; by proving a first-order statement in some nonstandard model (which may be easier than proving it in **R**), we know that the same statement must also be true of **R**.

The field **R** of real numbers is an extension field of the field **Q** of rational numbers, and **R** can therefore be seen as a vector space over **Q**. Zermelo–Fraenkel set theory with the axiom of choice guarantees the existence of a basis of this vector space: there exists a set B of real numbers such that every real number can be written uniquely as a finite linear combination of elements of this set, using rational coefficients only, and such that no element of B is a rational linear combination of the others. However, this existence theorem is purely theoretical; as such a base has never been explicitly described.

The well-ordering theorem implies that the real numbers can be well-ordered if the axiom of choice is assumed: there exists a total order on **R** with the property that every non-empty subset of **R** has a least element in this ordering. (The standard ordering ≤ of the real numbers is not a well-ordering since e.g. an open interval does not contain a least element in this ordering.) Again, the existence of such a well-ordering is purely theoretical, as it has not been explicitly described. If V=L is assumed in addition to the axioms of ZF, a well ordering of the real numbers can be shown to be explicitly definable by a formula

A real number may be either computable or un-computable; either algorithmically random or not; and either arithmetically random or not.

Check your Progress-1

1. Discuss the Number Series

____

____

2. Discuss about Compact sets

____

____

____

1.4 APPLICATIONS AND CONNECTIONS TO OTHER AREAS

1.4.1 Real Numbers and Logic

The real numbers are most often formalized using the Zermelo–Fraenkel axiomatization of set theory, but some mathematicians study the real numbers with other logical foundations of mathematics. In particular, the real numbers are also studied in reverse mathematics and in constructive mathematics.[11]

The hyper real as developed by Edwin Hewitt, Abraham Robinson and others extend the set of the real numbers by introducing infinitesimal and infinite numbers, allowing for building infinitesimal calculus in a way closer to the original intuitions of Leibniz, Euler, Cauchy and others.

Edward Nelson's internal set theory enriches the Zermelo–Fraenkel set theory syntactically by introducing a unary predicate "standard". In this approach, infinitesimals are (non-"standard") elements of the set of the real numbers (rather than being elements of an extension thereof, as in Robinson's theory).

The continuum hypothesis posits that the cardinality of the set of the real numbers is C, i.e. the smallest infinite cardinal number after N_0 , the cardinality of the integers. Paul Cohen proved in 1963 that it is an axiom independent of the other axioms of set theory; that is: one may choose either the continuum hypothesis or its negation as an axiom of set theory, without contradiction.

1.4.2 In physics

In the physical sciences, most physical constants such as the universal gravitational constant, and physical variables, such as position, mass, speed, and electric charge, are modelled using real numbers. In fact, the fundamental physical theories such as classical mechanics, electromagnetism, quantum mechanics, general relativity and the standard model are described using mathematical structures, typically smooth manifolds or Hilbert spaces, that are based on the real numbers, although actual measurements of physical quantities are of finite accuracy and precision.

Physicists have occasionally suggested that a more fundamental theory would replace the real numbers with quantities that do not form a continuum, but such proposals remain speculative.

1.4.3 In computation

With some exceptions, most calculators do not operate on real numbers. Instead, they work with finite-precision approximations called floating-point numbers. In fact, most scientific computation uses floating-point arithmetic. Real numbers satisfy the usual rules of arithmetic, but floating-point numbers do not.

Computers cannot directly store arbitrary real numbers with infinitely many digits. The achievable precision is limited by the number of bits allocated to store a number, whether as floating-pointnumbers or arbitrary-precision numbers. However, computer algebra systems can operate on irrational quantities exactly by manipulating formulas for them (such as $\sqrt{2}$, $\arcsin(2/23)$ or $\int_0^1 x^x \, dx$) rather than their rational or decimal approximation. It is not in general possible to determine whether two such expressions are equal (the constant problem).

A real number is called computable if there exists an algorithm that yields its digits. Because there are only accountably many algorithms but an uncountable number of real's, almost all real numbers fail to be computable. Moreover, the equality of two computable numbers is an undesirable problem. Some constructivists accept the existence of only those real's that are computable. The set of definable numbers is broader, but still only countable.

1.4.4 Real's in set theory

In set theory, specifically descriptive set theory, the Baire space is used as a surrogate for the real numbers since the latter have some topological properties (connectedness) that are a technical inconvenience. Elements of Baire space are referred to as "real’s".

1.5VOCABULARY AND NOTATION

1. Mathematicians use the symbol **R**, or, alternatively, ℝ, the letter "R" in blackboard bold to represent the set of all real numbers. As this set is naturally endowed with the structure of a field, the expression field of real numbers is frequently used when its algebraic properties are under consideration.

2 The sets of positive real numbers and negative real numbers are often noted $\mathbf{R}^+$ and $\mathbf{R}^-$, respectively; $\mathbf{R}_+$ and $\mathbf{R}_-$ are also used The non-negative real numbers can be noted $\mathbf{R}_{\geq 0}$ but one often sees this set noted $\mathbf{R}^+ \cup \{0\}$ In French mathematics, the positive real numbers and negative real numbers commonly include zero, and these sets are noted respectively $\mathbb{R}_+$ and $\mathbb{R}_-$. In this understanding, the respective sets without zero are called strictly positive real numbers and strictly negative real numbers, and are noted $\mathbb{R}_+$* and $\mathbb{R}_-$*.

3 The notation $\mathbf{R}^n$ refers to the Cartesian product of n copies of **R**, which is an n-dimensional vector space over the field of the real numbers; this vector space may be identified to the n-dimensional space of Euclidean geometry as soon as a coordinate system has been chosen in the latter. For example, a value from $\mathbf{R}^3$ consists of three real numbers and specifies the coordinates of a point in 3-dimensional space.

4 In mathematics, real is used as an adjective, meaning that the underlying field is the field of the real numbers (or the real field). For example, real matrix, real polynomial and real Lie algebra. The word is also used as a noun, meaning a real number (as in "the set of all real’s").

1.6 GENERALIZATIONS AND EXTENSIONS

The real numbers can be generalized and extended in several different directions:

- The complex numbers contain solutions to all polynomial equations and hence are an algebraically closed field unlike the real numbers. However, the complex numbers are not an ordered field.
- The affinely extended real number system adds two elements $+\infty$ and $-\infty$. It is a compact space. It is no longer a field, or even an additive group, but it still has a total order; moreover, it is a complete lattice.
- The real projective line adds only one value ∞. It is also a compact space. Again, it is no longer a field, or even an additive group. However, it allows division of a non-zero element by zero. It has cyclic order described by a separation relation.
- The long real line pastes together $\aleph_1^* + \aleph_1$ copies of the real line plus a single point (here $\aleph_1^*$ denotes the reversed ordering of $\aleph_1$) to create an ordered set that is "locally" identical to the real numbers, but somehow longer; for instance, there is an order-preserving embedding of $\aleph_1$ in the long real line but not in the real numbers. The long real line is the largest ordered set that is complete and locally Archimedean. As with the previous two examples, this set is no longer a field or additive group.
- Ordered fields extending the real's are the hyper real and the surreal numbers; both of them contain infinitesimal and infinitely large numbers and are therefore non-Archimedean ordered fields.
- Self-ad joint operators on a Hilbert space (for example, self-ad joint square complex matrices) generalize the real's in many respects: they can be ordered (though not totally ordered), they are complete, all their eigenvalues are real and they form a real associative algebra. Positive-definite operators correspond to the positive real's and normal operators correspond to the complex numbers

Continued Fraction

In mathematics, a **continued fraction** is an expression obtained through an iterative process of representing a number as the sum of its integer part and the reciprocal of another number, then writing this other number as the sum of its integer part and another reciprocal, and so on.[1] In a **finite continued fraction** (or **terminated continued fraction**), the iteration/recursion is terminated after finitely many steps by using an integer in lieu of another continued fraction. In contrast, an **infinite continued fraction** is an infinite expression. In either case, all integers in the sequence, other than the first, must be positive. The integers are called the coefficients or terms of the continued fraction.

Continued fractions have a number of remarkable properties related to the Euclidean algorithm for integers or real numbers. Every rational number / has two closely related expressions as a finite continued fraction, whose coefficients a_i can be determined by applying the Euclidean algorithm to (p,q). The numerical value of an infinite continued fraction is irrational; it is defined from its infinite sequence of integers as the limit of a sequence of values for finite continued fractions. Each finite continued fraction of the sequence is obtained by using a finite prefix of the infinite continued fraction's defining sequence of integers. Moreover, every irrational number Q is the value of a unique infinite continued fraction, whose coefficients can be found using the non-terminating version of the Euclidean algorithm applied to the incommensurable values ∞ and 1. This way of expressing real numbers (rational and irrational) is called their continued fraction representation.

It is generally assumed that the numerator of all of the fractions is 1. If arbitrary values and/or functions are used in place of one or more of the numerators or the integers in the denominators, the resulting expression is a **generalized continued fraction**. When it is necessary to distinguish the first form from generalized continued fractions, the former may be called a simple or regular continued fraction, or said to be in canonical form.

1.7 MOTIVATION AND NOTATION

If the starting number is rational, then this process exactly parallels the Euclidean algorithm. In particular, it must terminate and produce a finite continued fraction representation of the number. If the starting number is irrational, then the process continues indefinitely. This produces a sequence of approximations, all of which are rational numbers, and these converge to the starting number as a limit. This is the (infinite) continued fraction representation of the number. Examples of continued fraction representations of irrational numbers are:

- $\sqrt{19}$= [4;2,1,3,1,2,8,2,1,3,1,2,8,...] (sequence A010124 in the OEIS). The pattern repeats indefinitely with a period of 6.
- e = [2;1,2,1,1,4,1,1,6,1,1,8,...] (sequence A003417 in the OEIS). The pattern repeats indefinitely with a period of 3 except that 2 are added to one of the terms in each cycle.
- π = [3;7,15,1,292,1,1,1,2,1,3,1,...] (sequence A001203 in the OEIS). No pattern has ever been found in this representation.
- ϕ = [1;1,1,1,1,1,1,1,1,1,1,1,...] (sequence A000012 in the OEIS). The golden ratio, the irrational number that is the "most difficult" to approximate rationally. See: A property of the golden ratio φ.

Continued fractions are, in some ways, more "mathematically natural" representations of a real number than other representations such as decimal representations, and they have several desirable properties:

- The continued fraction representation for a rational number is finite

  and only rational numbers have finite representations. In contrast, the decimal

- Representation of a rational number may be finite, for example 137/1600 = 0.085625 or infinite with a repeating cycle, for example 4/27 = 0.148148148148...
- Every rational number has an essentially unique continued fraction representation. Each rational can be represented in exactly two ways, since [a_0; a1, a_{n-1}, an] = [a_0; a1, a_{n-1}, (a_n−1), 1]. Usually the first, shorter one is chosen as the canonical representation.

- The continued fraction representation of an irrational number is unique.
- The real numbers whose continued fraction eventually repeats are precisely the quadratic irrationals. For example, the repeating continued fraction [1;1,1,1,...] is the golden ratio, and the repeating continued
- Fraction [1; 2, 2, 2,] is the square root of 2. In contrast, the decimal representations of quadratic irrationals are apparently random. The square roots of all (positive) integers, that are not perfect squares, are quadratic irrationals, hence are unique periodic continued fractions.
- The successive approximations generated in finding the continued fraction
- Representation of a number, that is, by truncating the continued fraction representation, are in a certain sense (described below) the "best possible".

Basic formula

A continued fraction is an expression of the form

Where a_i and b_i can be any complex numbers. Usually they are required to be integers. If $b_i = 1$ for all i the expression is called a simple continued fraction. If the expression contains a finite number of terms, it is called a finite continued fraction. If the expression contains an infinite number of terms, it is called an infinite continued fraction.

Thus, all of the following illustrate valid finite simple continued fractions:

Examples of finite simple continued fractions		
Formula	**Numeric**	**Remarks**
a_0	2	All integers are a degenerate case

$a_0 + \frac{1}{a_1}$	$3+\frac{1}{4}$	Simplest possible fractional form
$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2}}$	$-3+\cfrac{1}{4+\cfrac{1}{8}}$	First integer may be negative
$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3}}}$	$\cfrac{1}{2+\cfrac{1}{5+\cfrac{1}{6}}}$	First integer may be zero

Calculating continued fraction representations

Consider a real number r. Let $i = \lfloor r \rfloor$ be the integer part of r and

Let f=r-i is the fractional part of r. Then the continued fraction representation of r is [i, a_1, a_2, a_3] where [a_1, a_2, a_3] is the continued fraction representation of 1/f.

To calculate a continued fraction representation of a number r, write down the integer part (technically the floor) of r. Subtract this integer part from r. If the difference is 0, stop; otherwise find the reciprocal of the difference and repeat. The procedure will halt if and only if r is rational. This process can be efficiently implemented using the Euclidean algorithm when the number is rational. The table below shows an implementation of this procedure for the number 3.245, resulting in the continued fraction expansion [3; 4, 12, 4].

Construction of the real numbers

The theorems of real analysis rely intimately upon the structure of the real number line. The real number system consists of a set (R), together with two binary operations denoted +and ·, and an order denoted <. The operations make the real numbers a field, and, along with the order, an ordered field. The real number system is the unique complete ordered field, in the sense that any other complete ordered field is isomorphic to it. Intuitively, completeness means that there are no 'gaps' in the real numbers. In particular, this property distinguishes the real numbers from other ordered fields (e.g., the rational numbers Q) and is critical to the proof of several key properties of functions of the real numbers. The completeness of the real’s is often conveniently expressed as the least upper bound property.

There are several ways of formalizing the definition of the real numbers. Modern approaches consist of providing a list of axioms, and a proof of the existence of a model for them, which has above properties. Moreover, one may show that any two models are isomorphic, which means that all models have exactly the same properties, and that one may forget how the model is constructed for using real numbers. Some of these constructions are described in the main article.

Order properties of the real numbers

The real numbers have several important lattice-theoretic properties that are absent in the complex numbers. Most importantly, the real numbers form an ordered field, in which sums and products of positive numbers are also positive. Moreover, the ordering of the real numbers is total, and the real numbers have the least upper bound property:

Every nonempty subset of that has an upper bound has a least upper bound that is also a real number.

These order-theoretic properties lead to a number of important results in real analysis, such as the monotone convergence theorem, the intermediate value theorem and the mean value theorem.

However, while the results in real analysis are stated for real numbers, many of these results can be generalized to other mathematical objects. In particular, many ideas in functional analysis and operator theory generalize properties of the real numbers – such generalizations include the theories of Riesz spaces and positive operators. Also, mathematicians consider real and imaginary parts of complex sequences, or by point wise evaluation of operator sequences.

Topological properties of the real numbers

Many of the theorems of real analysis are consequences of the topological properties of the real number line. The order properties of the real numbers described above are closely related to these topological properties. As a topological space, the real numbers has a standard topology, which is the order topology induced by order. Alternatively, by defining the metric or distance function using the absolute value function as, the real numbers become the prototypical example of a metric space. The topology induced by metric turns out to be identical to the standard topology induced by order. Theorems like the intermediate value theorem that are essentially topological in nature can often be proved in the more general setting of metric or topological spaces rather than in only. Often, such proofs tend to be shorter or simpler compared to classical proofs that apply direct methods.

Sequences

A **sequence** is a function whose domain is a countable, totally ordered set. The domain is usually taken to be the natural numbers, although it is occasionally convenient to also consider bidirectional sequences indexed by the set of all integers, including negative indices.

Of interest in real analysis, a **real-valued sequence**, here indexed by the natural numbers, is a map $a:N \to R, n \to a_n$. Each is referred to as a **term** (or, less commonly, an **element**) of the sequence. A sequence is rarely denoted explicitly as a function; instead, by convention, it is almost always notated as if it were an ordered ∞-tuple, with individual terms or a general term enclosed in parentheses:

$$(a_n) = (a_n)_{n \in N} = (a_1, a_2, a_3)$$

A sequence that tends to a limit (i.e., $\lim_{n\to\infty} a_n$ exists) is said to be **convergent**; otherwise it is **divergent**. (See the section on limits and convergence for details.) A real-valued sequence (a_n) is **bounded** if there exists $M \in R$ such that $|a_n| < M$ for all $n \in N$ .A real-valued sequence (a_n) is **monotonically increasing** or **decreasing** if

Notes

$a_1 \le a_2 \le a_3$ *or* $a_1 \ge a_2 \ge a_3$ Holds, respectively. If either holds, the sequence is said to be **monotonic**. The monotonicity is **strict** if the chained inequalities still hold with $\le$ or $\ge$ replaced by < or >.

Given a sequence (a_n) another sequence (b_k) is a **subsequence** of a_n if $b_k = a_{n_k}$ for all positive integers k and n_k is a strictly increasing sequence of natural numbers.

Check your Progress-2

1. Write the topological properties of real numbers

____

____

____

2. Write the definition of sequence and its function

____

____

____

Limits and convergence

Roughly speaking, a **limit** is the value that a function or a sequence "approaches" as the input or index approaches some value. (This value can include the symbols when addressing the behaviour of a function or sequence as the variable increases or decreases without bound.) The idea of a limit is fundamental to calculus (and mathematical analysis in general) and its formal definition is used in turn to define notions like continuity, derivatives, and integrals. (In fact, the study of limiting behaviour has been used as a characteristic that distinguishes calculus and mathematical analysis from other branches of mathematics.)

The concept of limit was informally introduced for functions by Newton and Leibniz, at the end of 17th century, for building infinitesimal calculus. For sequences, the concept was introduced by Cauchy, and made rigorous, at the end of 19th century by Bolzano and Weierstrass, who gave the modern ε-δ definition, which follows.

Definition. Let f be a real-valued function defined on an interval $E \subseteq R$. We say that f(x) **tends to L as x approaches x_0**, or that **the limit of f(x) as approaches x_0 is L** if, for any $\in > 0$, there exists $\delta > 0$ such that for all $x \in E, 0 < |x - x_0| < \delta$, implies that $|f(x) - L| < \in$. We write this symbolically as $f(x) \to L \; as \; x \to x_0, or \; \lim x \to x_0 f(x) = L$.

Intuitively, this definition can be thought of in the following way: We say that $f(x) \to L \; as \; x \to x_0$. when, given any positive number $\in$, no matter how small, we can always find a δ, such that we can guarantee that f(x) and L are less than $\in$ apart, as long as x (in the domain of) is a real number that is less than δ away from x_0 but distinct from x_0. The purpose of the last stipulation, which corresponds to the condition $0 < |x - x_0|$ in the definition, is to ensure that $\lim x \to x_0 f(x) = L$ does not imply anything about the value of $f(x_0)$ itself. Actually x_0, does not even need to be in the domain of f in order for $\lim x \to x_0 f(x)$ to exist.

In a slightly different but related context, the concept of a limit applies to the behaviour of a sequence (a_n) when n becomes large.

Definition. Let (a_n)be a real-valued sequence. We say that (a_n)**converges to** a if for any $\in > 0$ there exists a natural number N such that $n \geq N$ implies that $|a - a_n| < \in$. We write this symbolically as $a_n \to a as n \to \infty or \lim n \to \infty a_n = n$

If (a_n) fails to converge, we say that (a_n) **diverges**.

Generalizing to a real-valued function of a real variable, a slight modification of this definition (replacement of sequence $(a_n)$and term a_n by function f and value f(x) and natural numbers N and n by real numbers M and x, respectively) yields the definition of the **limit of f(x) as increases without bound**, notated . Reversing the inequality to gives the corresponding definition of the limit of as decreases without bound, $\lim x \to \infty f(x)$.

Sometimes, it is useful to conclude that a sequence converges, even though the value to which it converges is unknown or irrelevant. In these cases, the concept of a Cauchy sequence is useful.

Definition: Let (a_n) be real valued sequence. We say that (a_n) is a Cauchy sequence, if any $\in > 0$, there exists a natural number N such that $m, n \geq N$ implies that $|a_m - a_n| < \in$

It can be shown that a real-valued sequence is Cauchy if and only if it is convergent. This property of the real numbers is expressed by saying that the real numbers endowed with the standard metric $(R, |.|)$, is a **complete metric space**. In a general metric space, however, a Cauchy sequence need not converge.

In addition, for real-valued sequences that are monotonic, it can be shown that the sequence is bounded if and only if it is convergent.

Uniform and pointwise convergence for sequences of functions

In addition to sequences of numbers, one may also speak of sequences of functions on $E \subseteq R$, that is, infinite, ordered families of functions $f(n): E \subseteq R$, denoted $f(n)_{n=1}^{\infty}$, and their convergence properties. However, in the case of sequences of functions there are two kinds of convergence, known as point wise convergence and uniform convergence that need not be distinguished.

Roughly speaking, point wise convergence of functions f_n to a limiting a function $f(n): E \to R$ denoted by $f(n) \to f$,simply means that given any $x \in E, f_n(x) \to f(x) as\ n \to \infty$. In contrast, uniform convergence is a stronger type of convergence, in the sense that a uniformly convergent sequence of functions also converges point wise, but not conversely. Uniform convergence requires members of the family of functions f_n, to fall within some error $\in > 0$ of f for every value of $x \in E$, whenever $n \geq N$, for some integer N . For a family of functions to uniformly converge, sometimes denoted $f_n \Rightarrow f$, such a value of N must exist for any $\in > 0$ given , no matter how small. Intuitively, we can visualize this situation by imagining that, for a large enough N, the functions $f_n, f_{n+1}, f_{n+2}, \ldots$ are all confined within a 'tube' of width $2\in$

about f . for every value in their domain E. .

The distinction between point-wise and uniform convergence is important when exchanging the order of two limiting operations (e.g., taking a limit, a derivative, or integral) is desired: in order for the exchange to be well-behaved, many theorems of real analysis call for uniform convergence. For example, a sequence of continuous functions (see below) is guaranteed to converge to a continuous limiting function if the convergence is uniform, while the limiting function may not be continuous if convergence is only point wise. Karl Weierstrass is generally credited for clearly defining the concept of uniform convergence and fully investigating its implications.

There are two kinds of convergence, known as point wise convergence and uniform convergence that need to be distinguished.

Roughly speaking, point wise convergence of functions to a limiting function , denoted , simply means that given any , as . In contrast, uniform convergence is a stronger type of convergence, in the sense that a uniformly convergent sequence of functions also converges

Compactness

Compactness is a concept from general topology that plays an important role in many of the theorems of real analysis. The property of compactness is a generalization of the notion of a set being closed and bounded. (In the context of real analysis, these notions are equivalent: a set in Euclidean space is compact if and only if it is closed and bounded.) Briefly, a closed set contains all of its boundary points, while a set is bounded if there exists a real number such that the distance between any two points of the set is less than that number. In R , sets that are closed and bounded, and therefore compact, include the empty set, any finite number of points, closed intervals, and their finite unions. However, this list is not exhaustive; for instance, the set $\left\{1/n : n \in N\right\} \cup \{0\}$ is the compact set; the cantor ternary set $c \subset [0,1]$ is another example of a compact set. On the other hand, the set $\left\{1/n : n \in N\right\}$ is not compact because it is bounded but not closed, as the boundary point 0 is not a member of the set. The set $[0,\infty)$ is also not compact because it is closed but not bounded.

For subsets of the real numbers, there are several equivalent definitions of compactness.

Definition. A set $E \in R$ is compact if it is closed and bounded.

This definition also holds for Euclidean space of any finite dimension R^n, but it is not valid for metric spaces in general. The equivalence of the definition with the definition of compactness based on sub covers, given later in this section, is known as the Heine-Borel theorem.

A more general definition that applies to all metric spaces uses the notion of a subsequence (see above).

Definition. A set E in a metric space is compact if every sequence in E has a convergent subsequence.

This particular property is known as sub sequential compactness. In R, a set is sub sequentially compact if and only if it is closed and bounded, making this definition equivalent to the one given above. Sub sequential compactness is equivalent to the definition of compactness based on sub covers for metric spaces, but not for topological spaces in general.

The most general definition of compactness relies on the notion of open covers and sub covers, which is applicable to topological spaces (and thus to metric spaces and R as special cases). In brief, a collection of open sets U_∞ is said to be an open cover of set X if the union of these sets is a superset of X. This open cover is said to have a finite sub cover if a finite sub collection of the U_∞ could be found that also covers X.

Definition. A set X in a topological space is compact if every open cover of X has a finite sub cover.

Compact sets are well-behaved with respect to properties like convergence and continuity. For instance, any Cauchy sequence in a compact metric space is convergent. As another example, the image of a compact metric space under a continuous map is also compact.

Check your Progress-3

1. Discuss the limits and convergence

____

____

____

2. Discuss the uniform and point wise convergence

____

____

____

1.8 LETS SUM UP

In mathematics, a **real number** is a value of a continuous quantity that can represent a distance along a line. The adjective *real* in this context was introduced in the 17th century by René Descartes, who distinguished between real and imaginary roots of polynomials. The real numbers include all the rational numbers, such as the integer −5 and the fraction 4/3, and all the irrational numbers, such as √2 (1.41421356..., the square root of 2, an irrational algebraic number). Included within the irrationals are the transcendental numbers, such as π (3.14159265...). In addition to measuring distance, real numbers can be used to measure quantities such as time, mass, energy, velocity, and many more.

1.9 KEYWORD

Compactness

Compact

Real

Space

Limit

Convergence

Series

Sequence

Point wise

1.10 QUESTION FOR REVIEW

1. Give the Definition of Real Numbers.
2. What is the compact space?
3. Give the definition of topology space.
4. What is compactness?
5. Give some example of uniform and point wise convergence.
6. What are sequences?
7. What is continuous fraction?
8. Definition of Limit.
9. Give the definition of convergence with example.

1.11 REFERRENCE FOR FURTHER READING:

- *Siebeck, H. (1846). "Ueber periodische Kettenbrüche". J. Reine Angew. Math. 33. pp. 68–70.*
- *Heilermann, J. B. H. (1846). "Ueber die Verwandlung von Reihen in Kettenbrüche". J. Reine Angew. Math. 33. pp. 174–188.*
- *Magnus, Arne (1962). "Continued fractions associated with the Padé Table". Math. Z. 78. pp. 361–374.*

- *Chen, Chen-Fan; Shieh, Leang-San (1969). "Continued fraction inversion by Routh's Algorithm". IEEE Trans. Circuit Theory. 16 (2). pp. 197–202. doi:10.1109/TCT.1969.1082925.*
- *Gragg, William B. (1974). "Matrix interpretations and applications of the continued fraction algorithm". Rocky Mount. J. Math. 4 (2). p. 213. doi:10.1216/RJM-1974-4-2-213.*
- *Jones, William B.; Thron, W. J. (1980). Continued Fractions: Analytic Theory and Applications. Encyclopedia of Mathematics and its Applications. 11. Reading. Massachusetts: Addison-Wesley Publishing Company. ISBN 0-201-13510-8.*
- *Khinchin, A. Ya. (1964) [Originally published in Russian, 1935]. Continued Fractions. University of Chicago Press. ISBN 0-486-69630-8.*
- *Long, Calvin T. (1972), Elementary Introduction to Number Theory (2nd ed.), Lexington: D. C. Heath and Company, LCCN 77-171950*
- *Perron, Oskar (1950). Die Lehre von den Kettenbrüchen. New York, NY: Chelsea Publishing Company.*
- *Pettofrezzo, Anthony J.; Byrkit, Donald R. (1970), Elements of Number Theory, Englewood Cliffs: Prentice Hall, LCCN 77-81766*
- *Rockett, Andrew M.; Szüsz, Peter (1992). Continued Fractions. World Scientific Press. ISBN 981-02-1047-7.*
- H. S. Wall, *Analytic Theory of Continued Fractions*, D. Van Nostrand Company, Inc., 1948 ISBN 0-8284-0207-8
- *Cuyt, A.; Brevik Petersen, V.; Verdonk, B.; Waadeland, H.; Jones, W. B. (2008). Handbook of Continued fractions for Special functions. Springer Verlag. ISBN 978-1-4020-6948-2.*
- *Rieger, G. J. (1982). "A new approach to the real numbers (motivated by continued fractions)". Abh. Braunschweig.Wiss. Ges. 33. pp. 205–217.*

1.12 ANSWERS TO CHECK YOUR PROGRESS

Check in progress 1

1. Hint check section 1.3
2. Hint check section 1.7

Check in progress 2

1. ans hint - 1.4.5 Topological Properties Of The Real Numbers

2 ans hint Sequences (22/23)

Check in progress 3

1. ans hint - 1.0.1 Limits And Convergence
2. ans hint - 1.4.5.1 Uniform And Pointwise Convergence For Sequences Of Functions

UNIT - 2 : BASIC SET THEORY

STRUCTURE

2.0 OBJECTIVES

After studying this unit, you should be able to:

- learn about Sets
- Understand what is sets
- Work with set theory
- Construction of sets
- Learn about basic set theory

2.1 INTRODUCTION

Set theory is a branch of mathematical logic that studies sets, which informally are collections of objects. Although any type of object can be collected into a set, set theory is applied most often to objects that are relevant to mathematics. The language of set theory can be used to define nearly all mathematical objects.

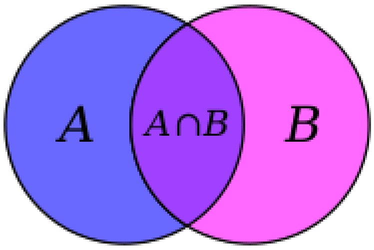


Fig. 1.1

The modern study of set theory was initiated by Georg Cantor and Richard Dedekind in the 1870s. After the discovery of paradoxes in naive set theory, such as Russell's paradox, numerous axiom systems were proposed in the early twentieth century, of which the Zermelo–Fraenkel axioms, with or without the axiom of choice, are the best-known.

Set theory is commonly employed as a foundational system for mathematics, particularly in the form of Zermelo–Fraenkel set theory with the axiom of choice. Beyond its foundational role, set theory is a branch of mathematics in its own right, with an active research community. Contemporary research into set theory includes a diverse collection of topics, ranging from the structure of the real number line to the study of the consistency of large cardinals.

2.2 HISTORY OF SETS

Mathematical topics typically emerge and evolve through interactions among many researchers. Set theory, however, was founded by a single paper in 1874 by Georg Cantor: "On a Property of the Collection of All Real Algebraic Numbers".

Since the 5th century BC, beginning with Greek mathematician Zeno of Elea in the West and early Indian mathematicians in the East, mathematicians had struggled with the concept of infinity. Especially notable is the work of Bernard Bolzano in the first half of the 19th century. Modern understanding of infinity began in 1870–1874 and was motivated by Cantor's work in real analysis.[4] An 1872 meeting between Cantor and Dedekind influenced Cantor's thinking and culminated in Cantor's 1874 paper.

Cantor's work initially polarized the mathematicians of his day. While Karl Weierstrass and Dedekind supported Cantor, Leopold Kronecker, now seen as a founder of mathematical constructivism, did not. Cantorian set theory eventually became widespread, due to the utility of Cantorian concepts, such as one-to-one correspondence among sets, his proof that there are more real numbers than integers, and the "infinity of infinities" ("Cantor's paradise") resulting from the power set operation. This utility of set theory led to the article "Mengenlehre" contributed in 1898 by Arthur Schoenflies to Klein's encyclopedia.

The next wave of excitement in set theory came around 1900, when it was discovered that some interpretations of Cantorian set theory gave rise to several contradictions, called antinomies or paradoxes. Bertrand Russell and Ernst Zermelo independently found the simplest and best known paradox, now called Russell's paradox: consider "the set of all sets that are not members of themselves", which leads to a contradiction since it must be a member of itself and not a member of itself. In 1899 Cantor had himself posed the question "What is the cardinal number of the set of all sets?", and obtained a related paradox. Russell used his paradox as a theme in his 1903 review of continental mathematics in his *The Principles of Mathematics*.

In 1906 English readers gained the book *Theory of Sets of Points* by husband and wife William Henry Young and Grace Chisholm Young, published by Cambridge University Press.

The momentum of set theory was such that debate on the paradoxes did not lead to its abandonment. The work of Zermelo in 1908 and the work of Abraham Fraenkel and Thoralf Skolem in 1922 resulted in the set of axioms ZFC, which became the most commonly used set of axioms for set theory. The work of analysts such as Henri Lebesgue demonstrated the great mathematical utility of set theory, which has since become woven into the fabric of modern mathematics. Set theory is commonly used as a foundational system, although in some areas—such as algebraic geometry and algebraic topology—category theory is thought to be a preferred foundation.

2.3 BASIC CONCEPTS AND NOTATION

Set theory begins with a fundamental binary relation between an object *o* and a set *A*. If *o* is a **member** (or **element**) of *A*, the notation $o \in A$ is used. A set is described by listing elements separated by commas, or by a characterizing property of its elements, within braces { }.Since sets are objects, the membership relation can relate sets as well.

A derived binary relation between two sets is the subset relation, also called **set inclusion**. If all the members of set *A* are also members of set *B*, then *A* is a **subset** of *B*, denoted $A \subseteq B$. For example, {1, 2} is a subset of {1, 2, 3} , and so is {2} but {1, 4} is not. As insinuated from this definition, a set is a subset of itself. For cases where this possibility is unsuitable or would make sense to be rejected, the term **proper subset** is defined. *A* is called a **proper subset** of *B* if and only if *A* is a subset of *B*, but *A* is not equal to *B*. Note also that 1, 2, and 3 are members (elements) of the set {1, 2, 3} but are not subsets of it; and in turn, the subsets, such as {1}, are not members of the set {1, 2, 3}.

Just as arithmetic features binary operations on numbers, set theory features binary operations on sets. The:

- **Union** of the sets *A* and *B*, denoted $A \cup B$, is the set of all objects that are a member of *A*, or *B*, or both. The union of {1, 2, 3} and {2, 3, 4} is the set {1, 2, 3, 4} .
- **Intersection** of the sets *A* and *B*, denoted $A \cap B$, is the set of all objects that are members of both *A* and *B*. The intersection of {1, 2, 3} and {2, 3, 4} is the set {2, 3} .
- **Set difference** of *U* and *A*, denoted $U \setminus A$, is the set of all members of *U* that are not members of *A*. The set difference {1, 2, 3} \ {2, 3, 4} is {1} , while, conversely, the set difference {2, 3, 4} \ {1, 2,

3} is {4} . When A is a subset of U, the set difference $U \setminus A$ is also called the **complement** of A in U. In this case, if the choice of U is clear from the context, the notation A^c is sometimes used instead of $U \setminus A$, particularly if U is a universal set as in the study of Venn diagrams.

- **Symmetric difference** of sets A and B, denoted $A \triangle B$ or $A \ominus B$, is the set of all objects that are a member of exactly one of A and B (elements which are in one of the sets, but not in both). For instance, for the sets {1, 2, 3} and {2, 3, 4} , the symmetric difference set is {1, 4} . It is the set difference of the union and the intersection, $(A \cup B) \setminus (A \cap B)$ or $(A \setminus B) \cup (B \setminus A)$.
- **Cartesian product** of A and B, denoted $A \times B$, is the set whose members are all possible ordered pairs (a, b) where a is a member of A and b is a member of B. The cartesian product of {1, 2} and {red, white} is {(1, red), (1, white), (2, red), (2, white)}.
- **Power set** of a set A is the set whose members are all of the possible subsets of A. For example, the power set of {1, 2} is { {}, {1}, {2}, {1, 2} } .

Some basic sets of central importance are the empty set (the unique set containing no elements; occasionally called the *null set* though this name is ambiguous), the set of natural numbers, and the set of real numbers.

In mathematics, a collection of particular things or group of particular objects is called a set. The theory of sets as developed George Cantor is being used in all branches of mathematics nowadays. According to him ‘A set is a well-defined collection of distinct objects of our perception or of our thought, to be conceived as a whole’.

As in the case of the concepts of geometrical point, line and a plane, a rigid definition is not possible for a set also. Is the intuitive conception of a collection or assemblage of things, real or conceptual.

The examples of the basic concepts of sets are:

(i) a set of living cricketers in the Australia.

(ii) a set of the rules for the badminton game;

(iii) a set of integers with prescribed conditions;

(iv) a set of books in the library;

(v) a set of the states in America;

Thus, the basic concept of sets is a well-defined collection of objects which are called members of the set or elements of the set. Objects belongs to the set must be well-distinguished.

Definition of set:

A set is a collection of well-defined objects.

Explanation of the term "Well-defined":

Well-defined means, it must be absolutely clear that which object belongs to the set and which does not.

For example:

'The collection of positive numbers less than 10' is a set, because, given any numbers, we can always find out whether that number belongs to the collection or not. But 'the collection of good students in your class' is not a set as in this case no definite rule is supplied by the help of which you can determine whether a particular student of your class is good or not. Thus, 'the collection of first five months of a year' is a set, but 'the collection of rich man in your town' is not a set.

Now, to get basic concepts of sets about the meaning of well-defined the following examples are given below.

1. The collection of vowels in English alphabets. This set contains five elements, namely, a, e, i, o, u.

2. A group of "Singers with ages between 18 years and 25 years" is a set, because the range of ages of the singer is given and so it can easily be decided that which singer is to be included and which is to be excluded. Hence, the objects are well-defined.

3. A collection of “Red flowers” is a set, because every red flowers will be included in this set i.e., the objects of the set are well-defined.

4. The collection of past presidents of the United States union is a set.

5. A group of “Young dancers” is not a set, as the range of the ages of young dancers is not given and so it can’t be decided that which dancer is to be considered young i.e., the objects are not well-defined.

6. The collection of cricketers in the world who were out for 99 runs in a test match is a set.

2.4AXIOMATIC SET THEORY

Elementary set theory can be studied informally and intuitively, and so can be taught in primary schools using Venn diagrams. The intuitive approach tacitly assumes that a set may be formed from the class of all objects satisfying any particular defining condition. This assumption gives rise to paradoxes, the simplest and best known of which are Russell's paradox and the Burali-Forti paradox. Axiomatic set theory was originally devised to rid set theory of such paradoxes.

The most widely studied systems of axiomatic set theory imply that all sets form a cumulative hierarchy. Such systems come in two flavors, those whose ontology consists of:

2.4.1 Sets Alone.

This includes the most common axiomatic set theory, Zermelo–Fraenkel set theory (ZFC), which includes the axiom of choice. Fragments of ZFC include:

- Zermelo set theory, which replaces the axiom schema of replacement with that of separation;
- General set theory, a small fragment of Zermelo set theory sufficient for the Peano axioms and finite sets;
- Kripke–Platek set theory, which omits the axioms of infinity, powerset, and choice, and weakens the axiom schemata of separation and replacement.

2.4.2 Sets and Proper Classes

These include Von Neumann–Bernays–Gödel set theory, which has the same strength as ZFC for theorems about sets alone, and Morse–Kelley set theory and Tarski–Grothendieck set theory, both of which are stronger than ZFC.

The above systems can be modified to allow urelements, objects that can be members of sets but that are not themselves sets and do not have any members.

The systems of New Foundations NFU (allowing urelements) and NF (lacking them) are not based on a cumulative hierarchy. NF and NFU include a "set of everything, " relative to which every set has a complement. In these systems urelements matter, because NF, but not NFU, produces sets for which the axiom of choice does not hold.

Systems of constructive set theory, such as CST, CZF, and IZF, embed their set axioms in intuitionist instead of classical logic. Yet other systems accept classical logic but feature a nonstandard membership relation. These include rough set theory and fuzzy set theory, in which the value of an atomic formula embodying the membership relation is not simply True or False. The Boolean-valued models of ZFC are a related subject.

2.5 APPLICATION

Many mathematical concepts can be defined precisely using only set theoretic concepts. For example, mathematical structures as diverse as graphs, manifolds, rings, and vector spaces can all be defined as sets satisfying various (axiomatic) properties. Equivalence and order relations are ubiquitous in mathematics, and the theory of mathematical relations can be described in set theory.

Set theory is also a promising foundational system for much of mathematics. Since the publication of the first volume of *Principia Mathematica*, it has been claimed that most or even all mathematical theorems can be derived using an aptly designed set of axioms for set theory, augmented with many definitions, using first or second-order logic. For example, properties of the natural and real numbers can be derived within set theory, as each number system can be identified with a set of equivalence classes under a suitable equivalence relation whose field is some infinite set.

Set theory as a foundation for mathematical analysis, topology, abstract algebra, and discrete mathematics is likewise uncontroversial; mathematicians accept that (in principle) theorems in these areas can be derived from the relevant definitions and the axioms of set theory. Few full derivations of complex mathematical theorems from set theory have been formally verified, however, because such formal derivations are often much longer than the natural language proofs mathematicians commonly present. One verification project, Metamath, includes human-written, computer-verified derivations of more than 12,000 theorems starting from ZFC set theory, first-order logic and propositional logic.

2.6 AREAS OF STUDY

Set theory is a major area of research in mathematics, with many interrelated subfields.

2.6.1 Combinatorial Set Theory

Combinatorial set theory concerns extensions of finite combinatorics to infinite sets. This includes the study of cardinal arithmetic and the study of extensions of Ramsey's theoremsuch as the Erdős–Rado theorem.

2.6.2 Descriptive Set Theory

Descriptive set theory is the study of subsets of the real line and, more generally, subsets of Polish spaces. It begins with the study of point classes in the Borel hierarchy and extends to the study of more complex hierarchies such as the projective hierarchy and the Wadge hierarchy. Many properties of Borel sets can be established in ZFC, but proving these properties hold for more complicated sets requires additional axioms related to determinacy and large cardinals.

The field of effective descriptive set theory is between set theory and recursion theory. It includes the study of lightface point classes, and is closely related to hyper arithmetical theory. In many cases, results of classical descriptive set theory have effective versions; in some cases, new results are obtained by proving the effective version first and then extending ("relativizing") it to make it more broadly applicable.

A recent area of research concerns Borel equivalence relations and more complicated definable equivalence relations. This has important applications to the study of invariants in many fields of mathematics. Descriptive set theory (DST) is the study of certain classes of "well-behaved" subsets of the real line and other Polish spaces. As well as being one of the primary areas of research in set theory, it has applications to other areas of mathematics such as functional analysis, ergodic theory, the study of operator algebras and group actions, and mathematical logic. Descriptive set theory begins with the study of Polish spaces and their Borel sets.

A Polish space is a second-countable topological space that is metrizable with a complete metric. Equivalently, it is a complete separable metric space whose metric has been "forgotten". Examples include the real line R, the Baire space N, the Cantor space C, and the Hilbert cube I . The class of Polish spaces has several universality properties, which show that there is no loss of generality in considering Polish spaces of certain restricted forms.

- Every Polish space is homeomorphic to a G_δ subspace of the Hilbert cube, and every G_δ subspace of the Hilbert cube is Polish.
- Every Polish space is obtained as a continuous image of Baire space; in fact every Polish space is the image of a continuous bijection defined on a closed subset of Baire space. Similarly, every compact Polish space is a continuous image of Cantor space.

Because of these universality properties, and because the Baire space has the convenient property that it is homeomorphic to , many results in descriptive set theory are proved in the context of Baire space alone.

The class of Borel sets of a topological space *X* consists of all sets in the smallest σ-algebra containing the open sets of *X*. This means that the Borel sets of *X* are the smallest collection of sets such that:

- Every open subset of X is a Borel set.
- If A is a Borel set, so is . That is, the class of Borel sets are closed under complementation.
- If A_n is a Borel set for each natural number n, then the union is a Borel set. That is, the Borel sets are closed under countable unions.

A fundamental result shows that any two uncountable Polish spaces X and Y are Borel isomorphic: there is a bijection from X to Y such that the preimage of any Borel set is Borel, and the image of any Borel set is Borel. This gives additional justification to the practice of restricting attention to Baire space and Cantor space, since these and any other Polish spaces are all isomorphic at the level of Borel sets.

Check your Progress:

Q. 1 Define Borel sets?

…………………………………………………………………………

……………………………………………………………………

Q .2 Define descriptive sets theory.

…………………………………………………………………………

…………………………………………………………………………

2.6.3 Fuzzy Set Theory

In set theory as Cantor defined and Zermelo and Fraenkel axiomatized, an object is either a member of a set or not. In fuzzy set theory this condition was relaxed by Lotfi A. Zadehso an object has a *degree of membership* in a set, a number between 0 and 1. For example, the degree of membership of a person in the set of "tall people" is more flexible than a simple yes or no answer and can be a real number such as 0.75.

Fuzzy sets (aka uncertain sets) are somewhat like sets whose elements have degrees of membership. Fuzzy sets were introduced independently by Lotfi A. Zadeh. and Dieter Klaua. in 1965 as an extension of the classical notion of set. At the same time, Salii (1965) defined a more general kind of structure called an L-relation, which he studied in an abstract algebraic context. Fuzzy relations, which are used now in different areas, such as linguistics (De Cock, Bodenhofer

& Kerre 2000), decision-making (Kuzmin 1982), and clustering (Bezdek 1978), are special cases of *L*-relations when *L* is the unit interval [0, 1].

In classical set theory, the membership of elements in a set is assessed in binary terms according to a bivalent condition — an element either belongs or does not belong to the set. By contrast, fuzzy set theory permits the gradual assessment of the membership of elements in a set; this is described with the aid of a membership function valued in the real unit interval [0, 1]. Fuzzy sets generalize classical sets, since the indicator functions (aka characteristic functions) of classical sets are special cases of the membership functions of fuzzy sets, if the latter only take values 0 or 1.[3] In fuzzy set theory, classical bivalent sets are usually called *crisp* sets. The fuzzy set theory can be used in a wide range of domains in which information is incomplete or imprecise, such as bioinformatics.

2.6.4 Inner Model Theory

An **inner model** of Zermelo–Fraenkel set theory (ZF) is a transitive class that includes all the ordinals and satisfies all the axioms of ZF. The canonical example is the constructible universe *L* developed by Gödel. One reason that the study of inner models is of interest is that it can be used to prove consistency results. For example, it can be shown that regardless of whether a model *V* of ZF satisfies the continuum hypothesis or the axiom of choice, the inner model *L* constructed inside the original model will satisfy both the generalized continuum hypothesis and the axiom of choice. Thus the assumption that ZF is consistent (has at least one model) implies that ZF together with these two principles is consistent.

The study of inner models is common in the study of determinacy and large cardinals, especially when considering axioms such as the axiom of determinacy that contradict the axiom of choice. Even if a fixed model of set theory satisfies the axiom of choice, it is possible for an inner model to fail to satisfy the axiom of choice. For example, the existence of sufficiently large cardinals implies that there is an inner model satisfying the axiom of determinacy (and thus not satisfying the axiom of choice).

Determinacy is a subfield of set theory, a branch of mathematics, that examines the conditions under which one or the other player of a game has a winning strategy, and the consequences of the existence of such strategies. Alternatively and similarly, "determinacy" is the property of a game whereby such a strategy exists.

The games studied in set theory are usually Gale–Stewart games – two-player games of perfect information in which the players make an infinite sequence of moves and there are no draws. The field of game theory studies more general kinds of games, including games with draws such as tic-tac-toe, chess, or infinite chess, or games with imperfect information such as poker.

Examples:-

- The class of all sets is an inner model containing all other inner models.
- The first non-trivial example of an inner model was the constructible universe *L* developed by Kurt Gödel. Every model *M* of ZF has an inner model L^{M} satisfying the axiom of constructability, and this will be the smallest inner model of *M* containing all the ordinals of *M*. Regardless of the properties of the original model, L^M will satisfy the generalized continuum hypothesis and combinatorial axioms such as the diamond principle ◊.
- The sets that are hereditarily ordinal definable form an inner model
- The sets that are hereditarily definable over a countable sequence of ordinals form an inner model, used in Solovay's theorem.

2.6.5 Large Cardinals

A **large cardinal** is a cardinal number with an extra property. Many such properties are studied, including inaccessible cardinals, measurable cardinals, and many more. These properties typically imply the cardinal number must be very large, with the existence of a cardinal with the specified property unprovable in Zermelo-Fraenkel set theory

A **large cardinal property** is a certain kind of property of transfinite cardinal numbers. Cardinals with such properties are, as the name suggests, generally very "large" (for example, bigger than the least α such that $\alpha=\omega_\alpha$). The proposition that such cardinals exist cannot be proved in the most common axiomatizationof set theory, namely ZFC, and such propositions can be viewed as ways of measuring how "much", beyond ZFC, one needs to assume to be able to prove certain desired results. In other words, they can be seen, in Dana Scott's phrase, as quantifying the fact "that if you want more you have to assume more".[1]

There is a rough convention that results provable from ZFC alone may be stated without hypotheses, but that if the proof requires other assumptions (such as the existence of large cardinals), these should be stated. Whether this is simply a linguistic convention, or something more, is a controversial point among distinct philosophical schools (see Motivations and epistemic status below).

A **large cardinal axiom** is an axiom stating that there exists a cardinal (or perhaps many of them) with some specified large cardinal property.

Most working set theorists believe that the large cardinal axioms that are currently being considered are consistent with ZFC. These axioms are strong enough to imply the consistency of ZFC. This has the consequence (via Gödel's second incompleteness theorem) that their consistency with ZFC cannot be proven in ZFC (assuming ZFC is consistent).

There is no generally agreed precise definition of what a large cardinal property is, though essentially everyone agrees that those in the list of large cardinal properties are large cardinal properties.

2.6.6 Determinacy

Determinacy refers to the fact that, under appropriate assumptions, certain two-player games of perfect information are determined from the start in the sense that one player must have a winning strategy. The existence of these strategies has important consequences in descriptive set theory, as the assumption that a broader class of games is determined often implies that a broader class of sets will have a topological property. The axiom of determinacy (AD) is an important object of study; although incompatible with the axiom of choice, AD implies that all subsets of the real line are well behaved (in particular, measurable and with the perfect set property). AD can be used to prove that the Wadge degreeshave an elegant structure.

Determinacy is a subfield of set theory, a branch of mathematics, that examines the conditions under which one or the other player of a game has a winning strategy, and the consequences of the existence of such strategies. Alternatively and similarly, "determinacy" is the property of a game whereby such a strategy exists.

The games studied in set theory are usually Gale–Stewart games – two-player games of perfect information in which the players make an infinite sequence of moves and there are no draws. The field of game theory studies more general kinds of games, including games with draws such as tic-tac-toe, chess, or infinite chess, or games with imperfect information such as poker.

The first sort of game we shall consider is the two-player game of perfect information of length ω, in which the players play natural numbers. These games are often called Gale–Stewart games.[1]

In this sort of game there are two players, often named *I* and *II*, who take turns playing natural numbers, with *I* going first. They play "forever"; that is, their plays are indexed by the natural numbers. When they're finished, a predetermined condition decides which player won. This condition need not be specified by any definable *rule*; it may simply be an arbitrary (infinitely long) lookup table saying who has won given a particular sequence of plays.

More formally, consider a subset *A* of Baire space; recall that the latter consists of all ω-sequences of natural numbers. Then in the game G_A, *I* plays a natural number a_0, then *II*plays a_1, then *I* plays a_2, and so on. Then *I* wins the game if and only if and otherwise *II* wins. *A* is then called the *payoff set* of G_A.

It is assumed that each player can see all moves preceding each of his moves, and also knows the winning condition.

All finite games of perfect information in which draws do not occur are determined.

Real-world games of perfect information, such as tic-tac-toe, chess, or infinite chess, are always finished in a finite number of moves (in chess-games this assumes the 50-move rule is applied). If such a game is modified so that a particular player wins under any condition where the game would have been called a draw, then it is always determined.[3] The condition that the game is always over (i.e. all possible extensions of the finite position result in a win for the same player) in a finite number of moves corresponds to the topological condition that the set *A* giving the winning condition for G_A is clopen in the topology of Baire space.

For example, modifying the rules of chess to make drawn games a win for Black makes chess a determined game.[4] As it happens, chess has a finite number of positions and a draw-by-repetition rules, so with these modified rules, if play continues long enough without White having won, then Black can eventually force a win (due to the modification of draw = win for black).

The proof that such games are determined is rather simple: Player *I* simply plays *not to lose*; that is, he plays to make sure that player *II* does not have a winning strategy *after I'*s move. If player *I cannot* do this, then it means player *II* had a winning strategy from the beginning. On the other hand, if player *I can* play in this way, then he must win, because the game will be over after some finite number of moves, and he can't have lost at that point.

This proof does not actually require that the game *always* be over in a finite number of moves, only that it be over in a finite number of moves whenever *II* wins. That condition, topologically, is that the set *A* is closed. This fact—that all closed games are determined—is called the *Gale–Stewart theorem*. Note that by symmetry, all open games are determined as well. (A game is *open* if *I* can win only by winning in a finite number of moves.)

2.6.7 Set-Theoretic Topology

Set-theoretic topology studies questions of general topology that are set-theoretic in nature or that require advanced methods of set theory for their solution. Many of these theorems are independent of ZFC, requiring stronger axioms for their proof. A famous problem is the normal Moore space question, a question in general topology that was the subject of intense research. The answer to the normal Moore space question was eventually proved to be independent of ZFC.

Example 1. In a competition, a school awarded medals in different categories. 36 medals in dance, 12 medals in dramatics and 18 medals in music. If these medals went to a total of 45 persons and only 4 persons got medals in all the three categories, how many received medals in exactly two of these categories?

Solution:

Let A = set of persons who got medals in dance.

B = set of persons who got medals in dramatics.

C = set of persons who got medals in music.

Given,

$n(A) = 36$,$n(B) = 12$, $n(C) = 18$

$n(A \cup B \cup C) = 45$,$n(A \cap B \cap C) = 4$

We know that number of elements belonging to exactly two of the three sets A, B, C

$= n(A \cap B) + n(B \cap C) + n(A \cap C) - 3n(A \cap B \cap C)$

$= n(A \cap B) + n(B \cap C) + n(A \cap C) - 3 \times 4$ ……..(i)

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

Therefore, $n(A \cap B) + n(B \cap C) + n(A \cap C) = n(A) + n(B) + n(C) + n(A \cap B \cap C) - n(A \cup B \cup C)$

From (i) required number

$= n(A) + n(B) + n(C) + n(A \cap B \cap C) - n(A \cup B \cup C) - 12$

$= 36 + 12 + 18 + 4 - 45 - 12$

$= 70 - 57$

$= 13$

Example 2. Each student in a class of 40 plays at least one indoor game chess, carrom and scrabble. 18 play chess, 20 play scrabble and 27 play carrom. 7 play chess and scrabble, 12 play scrabble and carrom and 4 play chess, carrom and scrabble. Find the number of students who play (i) chess and carrom. (ii) chess, carrom but not scrabble.

Notes

Solution:

Let A be the set of students who play chess

B be the set of students who play scrabble

C be the set of students who play carrom

Therefore, We are given $n(A \cup B \cup C) = 40$,

$n(A) = 18$, $n(B) = 20$ $n(C) = 27$,

$n(A \cap B) = 7$, $n(C \cap B) = 12$ $n(A \cap B \cap C) = 4$

We have

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$

Therefore, $40 = 18 + 20 + 27 - 7 - 12 - n(C \cap A) + 4$

$40 = 69 – 19 - n(C \cap A)$

$40 = 50 - n(C \cap A)$ $n(C \cap A) = 50 - 40$

$n(C \cap A) = 10$

Therefore, Number of students who play chess and carrom are 10.

Also, number of students who play chess, carrom and not scrabble.

$= n(C \cap A) - n(A \cap B \cap C)$

$= 10 – 4$

$= 6$

Therefore, we learned how to solve different types of word problems on sets without using Venn diagram.

2.7 LETS SUM UP

From set theory's inception, some mathematicians have objected to it as a foundation for mathematics. The most common objection to set theory, one Kronecker voiced in set theory's earliest years, starts from the constructivist view that mathematics is loosely related to computation. If this view is granted, then the treatment of infinite sets, both in naive and in axiomatic set theory, introduces into mathematics methods and objects that are not computable even in principle. The feasibility of constructivism as a substitute foundation for mathematics was greatly increased by Errett Bishop's influential book *Foundations of Constructive Analysis*

A different objection put forth by Henri Poincaré is that defining sets using the axiom schemas of specification and replacement, as well as the axiom of power set, introduces impredicativity, a type of circularity, into the definitions of mathematical objects. The scope of predicatively founded mathematics, while less than that of the commonly accepted Zermelo-Fraenkel theory, is much greater than that of constructive mathematics, to the point that Solomon Feferman has said that "all of scientifically applicable analysis can be developed [using predicative methods]".

Ludwig Wittgenstein condemned set theory. He wrote that "set theory is wrong", since it builds on the "nonsense" of fictitious symbolism, has "pernicious idioms", and that it is nonsensical to talk about "all numbers".Wittgenstein's views about the foundations of mathematics were later criticised by Georg Kreisel and Paul Bernays, and investigated by Crispin Wright, among others.

Category theorists have proposed topos theory as an alternative to traditional axiomatic set theory. Topos theory can interpret various alternatives to that theory, such as constructivism, finite set theory, and computable set theory. Topoi also give a natural setting for forcing and discussions of the independence of choice from ZF, as well as providing the framework for pointless topology and Stone spaces.

An active area of research is the univalent foundations and related to it homotopy type theory. Within homotopy type theory, a set may be regarded as a homotopy 0-type, with universal properties of sets arising from the inductive and recursive properties of higher inductive types. Principles such as the axiom of choice and the law of the excluded middle can be formulated in a manner corresponding to the classical formulation in set theory or perhaps in a spectrum of distinct ways unique to type theory. Some of these principles may be proven to be a consequence of other principles. The variety of formulations of these axiomatic principles allows for a detailed analysis of the formulations required in order to derive various mathematical results.

2.8 KEYWORD

CONSTRUCTIVE

AXIOMATIC

TOPOLOGY

SETS

VEN

CARDINAL

2.9 QUESTION FOR REVIEW

Q. 1 what is fuzzy sets?

Q. 2 Define basic sets in real analysis?

Q. 3 Define alone sets in real analysis?

Q. 4 what is Venn’s Diagram define with example.

Q. 5 Define the determinacy?

Q. 6 What is descriptive sets theory?

2.10 SUGGESTION READING & REFERENCES

1. *Cantor, Georg (1874), "Ueber eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen", Journal für die reine und angewandte Mathematik (in German),* ***77****: 258–262, doi:10.1515/crll.1874.77.258*
2. *^ Johnson, Philip (1972), A History of Set Theory, Prindle, Weber & Schmidt, ISBN 0-87150-154-6*
3. *^ Bolzano, Bernard (1975), Berg, Jan (ed.), Einleitung zur Größenlehre und erste Begriffe der allgemeinen Größenlehre, Bernard-Bolzano-Gesamtausgabe, edited by Eduard Winter et al., Vol. II, A, 7, Stuttgart, Bad Cannstatt: Friedrich Frommann Verlag, p. 152, ISBN 3-7728-0466-7*

4. ^ *Dauben, Joseph (1979), Georg Cantor: His Mathematics and Philosophy of the Infinite, Harvard University Press, pp. 30–54, ISBN 0-674-34871-0.*
5. ^ *Young, William; Young, Grace Chisholm (1906), Theory of Sets of Points, Cambridge University Press*
6. ^ *Kolmogorov, A.N.; Fomin, S.V. (1970), Introductory Real Analysis (Rev. English ed.), New York: Dover Publications, pp. 2–3, ISBN 0486612260, OCLC 1527264*
7. ^ *Jech, Thomas (2003), Set Theory, Springer Monographs in Mathematics (Third Millennium ed.), Berlin, New York: Springer-Verlag, p. 642, ISBN 978-3-540-44085-7, Zbl 1007.03002*
8. ^ *Bishop, Errett (1967), Foundations of Constructive Analysis, New York: Academic Press, ISBN 4-87187-714-0*
9. ^ *Feferman, Solomon (1998), In the Light of Logic, New York: Oxford University Press, pp. 280–283, 293–294, ISBN 0195080300*
10. ^ *Wittgenstein, Ludwig (1975), Philosophical Remarks, §129, §174, Oxford: Basil Blackwell, ISBN 0631191305*

2.11 ANSWERS TO CHECK YOUR PROGRESS

Check in Progress 1

1. Hint refer section 2.6
2. Hint please refer 2.6.2
3. Hint refer section 2.6

UNIT - 3:EXTENDED REAL NUMBERS

STRUCTURE

3.1 NEED OF REAL NUMBERS

This is a good juncture to justify the subject of real analysis, which essentially reduces to justifying the necessity of studying R. So, what is missing? Why do we need anything beyond the rational ?

The first sign of trouble is square roots. Famously, $\sqrt{2}$ is not rational – in other words, there is no rational number which squares to 2 this fact has a curious consequence – consider the following function:

$$f : \mathbb{Q} \to \mathbb{Q} \,;\, x \mapsto \begin{cases} 0 & : x^2 < 2 \\ 1 & : x^2 > 2 \end{cases}$$

Clearly this function has a dramatic jump in it around the rational, where it suddenly changes from being equal to zero and starts being equal to one. However, it's difficult (or even impossible) to pin down exactly where this jump happens. Any specific rational number is safely on one side or the other, and, indeed, in the standard Topology on Q, this function is continuous .It is this flaw which the real numbers are designed to repair. We will define the real numbers ℝ so that no matter how clever we try to be, if a function has a 'jump' in the way that F does, then we will always be able to find a specific number at which it jumps.

The following sections describe the properties of ℝ which make this possible.

3.2DIFFERENT PERSPECTIVES

In order to prove anything about the real numbers, we need to know what their properties are. There are two different approaches to describing these properties – axiomatic and constructive

3.2.1 An Axiomatic Approach

When we take an axiomatic approach, we simply make a series of assertions regarding ℝ , and assume that they hold.

The assertions that we make are called *axioms* – in a mathematical context this term means roughly 'basic assumption'.

The advantage of this approach is that it is then clear exactly what has been assumed, before proceeding to deduce results which rely only on those assumptions.

The disadvantage of this approach is that it might not be immediately clear that any object satisfying the properties we desire even exists!

3.2.2 A Constructive Approach

With a constructive approach, we are not happy simply to assume exactly what we want, but rather we try to *construct* ℝ from something simpler, and then prove that it has the properties we want. In this way, what could have been axioms become theorems. There are several different ways to do this, starting from $\mathbb{Q}$ and using some method to 'fill up the gaps between the rationals'.

All of these methods are fairly complex and will be put off until the next section.

So, what are these axioms which we will need? The short version is to say that is a *complete ordered field*. This is in fact saying a great many things:

- That $\mathbb{R}$ is a totally ordered field.
- That $\mathbb{R}$ is complete in this ordering (Note that the meaning of completeness here is not quite the same as the common meaning in the study of partially ordered sets).
- That the algebraic operations (addition and multiplication) described by the field axioms interact with the ordering in the expected manner. In more detail, we assert the following:
- $\mathbb{R}$ is a field For this, we require binary operations *addition (denoted as +*)and *multiplication (denoted as X)defined on* $\mathbb{R}$ and distinct elements defined as 0 and 1 are satisfying

1. $(\mathbb{R}, +, 0)$ is a commutative group, meaning:
    1. $\forall x, y, z \in \mathbb{R} : (x + y) + z = x + (y + z)$ (associativity)
    2. $\forall x, y \in \mathbb{R} : x + y = y + x$ (commutativity)
    3. $\forall x \in \mathbb{R} : x + 0 = x$ (identity)
    4. $\forall x \in \mathbb{R} : \exists y \in \mathbb{R} : x + y = 0$ (inverse)
2. $(\mathbb{R} \setminus \{0\}, \times, 1)$ is a commutative group, meaning:
    1. $\forall x, y, z \in \mathbb{R} \setminus \{0\} : (x \times y) \times z = x \times (y \times z)$ (associativity)
    2. $\forall x, y \in \mathbb{R} \setminus \{0\} : x \times y = y \times x$ (commutativity)
    3. $\forall x \in \mathbb{R} \setminus \{0\} : x \times 1 = x$ (identity)
    4. $\forall x \in \mathbb{R} \setminus \{0\} : \exists y \in \mathbb{R} \setminus \{0\} : x \times y = 1$ (inverse)
3. $\forall x, y, z \in \mathbb{R} : x \times (y + z) = (x \times y) + (x \times z)$ (distributivity)

2. $\mathbb{R}$ is a totally ordered set. For this we require a relation (denoted by $\leq$) satisfying:
    1. $\forall x \in \mathbb{R} : x \leq x$ (reflexivity)
    2. $\forall x, y, z \in \mathbb{R} : (x \leq y \text{ and } y \leq z) \implies x \leq z$ (transitivity)
    3. $\forall x, y \in \mathbb{R} : (x \leq y \text{ and } y \leq x) \implies x = y$ (anti-symmetry)
    4. $\forall x, y \in \mathbb{R} : \text{either } x \leq y \text{ or } y \leq x$ (totality)
3. $\mathbb{R}$ is complete in this order (see below for details).
4. The field operations and order interact in the expected manner, meaning:
    1. $\forall x, y, z \in \mathbb{R} : x \leq y \implies (x + z) \leq (y + z)$
    2. $\forall x, y, z \in \mathbb{R} : (x \leq y \text{ and } 0 \leq z) \implies (x \times z) \leq (y \times z)$

This is amongst the longest list of axioms in any region of mathematics, but if you examine each in turn, you will find that they all state things which you have probably taken for granted as 'the way numbers behave' without a second thought.

These axioms are so exacting that there is a sense in which they specify the real numbers precisely. In other words ℝ is the only complete ordered field.

Affinely Extended Real number system is obtained from the real number system ℝ by adding two elements: $+\infty$ and $-\infty$ (read as **positive infinity** and **negative infinity** respectively). These new elements are not real numbers. It is useful in describing various limiting behaviors in calculus and mathematical analysis, especially in the theory of measure and integration. The affinely extended real number system is denoted or $[-\infty, +\infty]$ or $\mathbb{R} \cup \{-\infty, +\infty\}$.

When the meaning is clear from context, the symbol $+\infty$ is often written simply as ∞.

3.2.3 Limits

We often wish to describe the behaviour of a function$f(X)$, as either the argument or the function value$f(X)$, gets "very big" in some sense. For example, consider the function. The graph of this function $f(X) = X^{-2}$ has a horizontal asymptote at y = 0. Geometrically, as we move farther and farther to the right along the x-axis, the value of $\frac{1}{x2}$ approaches 0. This limiting behaviour is similar to the limit of a function at a real number, except that there is no real number to which approaches.

By adjoining the elements $+\infty$ and $-\infty$, to R we allow a formulation of a "limit at infinity" with topological properties similar to those for to ℝ

To make things completely formal, the Cauchy sequences definition of ℝ allows $+\infty$ us to define as the set of all sequences of rationals which, for any K > 0, from some point on exceed K. We can define $-\infty$ similarly.

3.3 MEASURE AND INTEGRATION

Notes

In measure theory, it is often useful to allow sets that have infinite measure and integrals whose value may be infinite.

Such measures arise naturally out of calculus. For example, in assigning a measure to ℝ that agrees with the usual length of intervals, this measure must be larger than any finite real number. Also, when considering improper integrals, such as $\int_1^\infty \frac{dx}{x}$

the value "infinity" arises. Finally, it is often useful to consider the limit of a sequence of functions, such as.

$$Fn(x) = \begin{cases} 2n(1-nx)\ if\ 0 \le x \le 1/n \\ 0\ , \quad \frac{1}{n} < x \le 1 \end{cases}$$

Without allowing functions to take on infinite values, such essential results as the monotone convergence theorem and the dominated convergence theorem would not make sense.

3.3.1 Order And Topological Properties

The affinely extended real number system turns into a totally ordered set by defining -∞ ≤ α + ∞ for all α . This order has the desirable property that every subset has a supremum and an infimum: it is a complete lattice.

This induces the order topology on ℝ. In this topology, a set U is a neighbourhood of +∞ if and only if it contains a set {x : x > a } for some real number α, and analogously for the neighbourhoods-∞of R. is a compact Hausdorf space homeomorphic to the unit interval [0,1]. Thus the topology is metrizable, corresponding (for a given homeomorphism) to the ordinary metric on this interval. There is no metric that is an extension of the ordinary metric on ℝ.

With this topology +∞ the - ∞specially defined limits for tending to and , and the specially defined concepts of limits equal to and , reduce to the general topological definitions of limits.

3.3.2 Arithmetic Operations

The arithmetic operations of ℝ can be partially extended ℝ as follows: a + ∞ = + ∞ + a = + ∞ , a ≠ - ∞

a - ∞ = - ∞ + a = - ∞ , a ≠ + ∞

a.(±∞) = ± ∞.a = ± ∞, a € (0, +∞)

a.(±∞) = ± ∞.a = ± ∞, a € [- ∞, 0)

$\frac{a}{\pm\infty} = 0$, a € R

$$\frac{\pm\infty}{a} = \pm\infty, \quad a \in (0, +\infty)$$

$$\frac{\pm\infty}{a} = \mp\infty, \quad a \in (-\infty, 0)$$

For exponentiation, see Exponentiation Limits of powers. Here, "a +∞" means both "a + (+∞)" and "a – (-∞)", while "a -∞" means both "a – (+∞)" and "a + (-∞)".

The expressions ∞ -∞ and (called indeterminate forms) are usually left undefined. These rules are modelled on the laws for infinite limits. However, in the context of probability or measure theory, is often defined as .The expression $\frac{1}{0}$ is not defined either +∞ as or -∞ , because although it is true that whenever f (x) → 0 for a continuous function f(x) it must be the case that 1/ f(x) is eventually contained in every neighbourhood of the set{- ∞, + ∞} , it is *not* true that 1/ F(x) must tend to + ∞ or - ∞ one of these points. An example is which is of the form but does not tend to either or when . For instance, but does not exist because but . (The modulus , nevertheless, does approach .)

3.3.3 Algebraic Properties

With these definitions R is **not** even a semigroup, let alone a group, a ring or a field, like R is one. However, it still has several convenient properties:

- a + (b + c) and (a + b) + c are either equal or both undefined.
- a + b and b + a are either equal or both undefined.
- a. (b. c) and (a. b) . c are either equal or both undefined.
- a. b and b. a are either equal or both undefined
- a. (b + c) and a .b + a. c are equal if both are defined.
- a ≤ b If a + c and b + c if both and are defined, then a + c ≤ b + c .

- If a ≤ b and c > 0 and if both a .c and b. c are defined, then a. c ≤ b. c . In general, all laws of arithmetic are valid in R as long as all occurring expressions are defined.

Miscellaneous

Several functions can be continuously extended to $\overline{\mathbb{R}}$ by taking limits. For instance, one defines the extrermal points of the following functions as follows:

$$\exp(-\infty) = 0,\ \ln(0) = -\infty,\ \tanh(\pm\infty) = \pm 1,\ \arctan(\pm\infty) = \pm\frac{\pi}{2}$$

Some singularities may additionally be removed. For example, the function $\frac{1}{x^2}$ can be continuously extended to $\overline{\mathbb{R}}$ (under *some* definitions of continuity) by setting the value to + ∞ for x =0 , and 0 for x = + ∞ and x = -∞ . The function 1/x can*not* be continuously extended because the function approaches -∞ as approaches 0 from below, and + ∞ as approaches 0 from above.

Compare the projectively extended real line, which does not distinguish between +∞ and -∞ . As a result, on one hand a function may have limit ∞ on the projectively extended real line, while in the affinely extended real number system only the absolute value of the function has a limit, e.g. in the case of the function 1/x at x= 0 . On the other hand $\lim_{x \to -\infty} f(x)$ and $\lim_{x \to +\infty} f(x)$

correspond on the projectively extended real line to only a limit from the right and one from the left, respectively, with the full limit only existing when the two are equal. Thus e^x and **arctan(x)** cannot be made continuous at x=∞ on the projectively extended real line.

In real analysis, the **projectively extended real line** (also called the one-point compactification of the real line), is the extension of the number line by a point denoted ∞. It is thus the set $\mathbb{R} \cup \{\infty\}$ (where $\mathbb{R}$ is the set of the real numbers) with the standard arithmetic operations extended where possible, sometimes denoted by $\widehat{\mathbb{R}}$.The added point is called the point at infinity, because it is considered as a neighbour of both ends of the real line. More precisely, the point at infinity is the limit of every sequence of real numbers whose absolute values are increasing and unbounded.

The projectively extended real line may be identified with the projective line over the reals in which three points have been assigned specific values (e.g. 0, 1 and ∞). The projectively extended real line must not be confused with the extended real number line, in which +∞ and −∞are distinct.

Unlike most mathematical models of the intuitive concept of 'number', this structure allows division by zero:

$$\frac{a}{0} = \infty$$

for nonzero *a*. In particular 1/0 = ∞, and moreover 1/∞ = 0, making reciprocal, $1/x$, a total function in this structure. The structure, however, is not a field, and none of the binary arithmetic operations are total, as witnessed for example by $0 \cdot \infty$ being undefined despite the reciprocal being total. It has usable interpretations, however – for example, in geometry, a vertical line has *infinite* slope.

3.4. EXTENSIONS OF THE REAL LINE

The projectively extended real line extends the field of real numbers in the same way that the Riemann sphere extends the field of complex numbers, by adding a single point called conventionally ∞.

In contrast, the extended real number line (also called the two-point compactification of the real line) distinguishes between +∞ and -∞.The order relation cannot be extended to $\widehat{\mathbb{R}}$ in a meaningful way. Given a number $a \neq \infty$, there is no convincing argument to define $a>\infty$either or that $a<\infty$. Since ∞ can't be compared with any of the other elements, there's no point in retaining this relation on $\widehat{\mathbb{R}}$. However, order on $\mathbb{R}$ is used in definitions in $\widehat{\mathbb{R}}$.

3.4.1 Geometry

Fundamental to the idea that ∞ is a point *no different from any other* is the way the real projective line is a homogeneous space, in fact homeomorphic to a circle. For example the general linear group of

2×2 real invertible matrices has a transitive action on it. The group action may be expressed by Möbius transformations, (also called linear fractional transformations), with the understanding that when the denominator of the linear fractional transformation is 0, the image is ∞.

The detailed analysis of the action shows that for any three distinct points *P*, *Q* and *R*, there is a linear fractional transformation taking *P* to 0, *Q* to 1, and *R* to ∞ that is, the group of linear fractional transformations is triply transitive on the real projective line. This cannot be extended to 4-tuples of points, because the cross-ratio is invariant.

The terminology projective line is appropriate, because the points are in 1-to-1 correspondence with one-dimensional linear subspaces of $\mathbb{R}^2$

3.4.2 Arithmetic Operations

Motivation for arithmetic operations

The arithmetic operations on this space are an extension of the same operations on reals. A motivation for the new definitions is the limits of functions of real number.

Arithmetic operations that are defined

In addition to the standard operations on the subset $\mathbb{R}$ of $\widehat{\mathbb{R}}$, the following operations are defined for $a \in \widehat{\mathbb{R}}$, with exceptions as indicated:

$$\begin{aligned} a+\infty = \infty + a &= \infty, & a &\neq \infty \\ a-\infty = \infty - a &= \infty, & a &\neq \infty \\ a/\infty = a\cdot 0 = 0\cdot a &= 0, & a &\neq \infty \\ \infty/a &= \infty, & a &\neq \infty \\ a/0 = a\cdot\infty = \infty\cdot a &= \infty, & a &\neq 0 \\ 0/a &= 0, & a &\neq 0 \end{aligned}$$

Arithmetic operations that are left undefined.

The following expressions cannot be motivated by considering limits of real functions, and no definition of them allows the statement of the standard algebraic properties to be retained unchanged in form for all defined cases Consequently, they are left undefined:

$$\infty + \infty$$
$$\infty - \infty$$
$$\infty \cdot 0$$
$$0 \cdot \infty$$
$$\infty / \infty$$
$$0/0$$

Algebraic properties

The following equalities mean: *Either both sides are undefined, or both sides are defined and equal.* This is true for any a,b,c ∈ ℝ

$$\begin{aligned}(a+b)+c &= a+(b+c)\\ a+b &= b+a\\ (a\cdot b)\cdot c &= a\cdot(b\cdot c)\\ a\cdot b &= b\cdot a\\ a\cdot\infty &= \frac{a}{0}\end{aligned}$$

The following is true whenever the right-hand side is defined, for any a,b,c ∈ $\widehat{\mathbb{R}}$

$$\begin{aligned}a\cdot(b+c) &= a\cdot b + a\cdot c\\ a = \left(\frac{a}{b}\right)\cdot b &= \frac{(a\cdot b)}{b}\\ a = (a+b)-b &= (a-b)+b\end{aligned}$$

In general, all laws of arithmetic that are valid for are ℝ also valid for $\widehat{\mathbb{R}}$ whenever all the occurring expressions are defined.

Intervals and topology

The concept of an interval can be extended to $\widehat{\mathbb{R}}$. However, since it is an unordered set, the interval has a slightly different meaning. The definitions for closed intervals are as follows (it is assumed that a,b ∈ ℝ, a<b):

$$[a,b] = \{x / x \in R, a \le x \le b\}$$

$$[a,\infty] = \{x / x \in R, a \le x \le b\} \cup \{\infty\}$$

$$[b,a] = \{x / x \in R, b \le x\} \cup \{\infty\} \cup \{x / x \in R, x \le a\}$$

$$[a,a] = \{a\}$$

$$[\infty,\infty] = \{\infty\}$$

With the exception of when the end-points are equal, the corresponding open and half-open intervals are defined by removing the respective endpoints.

$\widehat{\mathbb{R}}$ and the empty set are each also an interval, as $\widehat{\mathbb{R}}$is excluding any single point.

The open intervals as base define a topology on $\widehat{\mathbb{R}}$. Sufficient for a base are the finite open intervals in $\mathbb{R}$ and the intervals

$(b,a) = \{x / x \in R, b < x\} \cup \{\infty\} \cup \{x / x \in R, x < a\}$ for all $a,b \in R \, such that \, a < b$

As said, the topology is homeomorphic to a circle. Thus it is metrizable corresponding (for a given homeomorphism) to the ordinary metric on this circle (either measured straight or along the circle). There is no metric which is an extension of the ordinary metric on $\mathbb{R}$.

Interval arithmetic

Interval arithmetic extends $\widehat{\mathbb{R}}$ to from $\mathbb{R}$. The result of an arithmetic operation on intervals is always an interval, except when the intervals with a binary operation contain incompatible values leading to an undefined result. In particular, we have, for every a,b ϵ $\widehat{\mathbb{R}}$:

$$x \in [a,b] \iff \frac{1}{x} \in \left[\frac{1}{b}, \frac{1}{a}\right],$$

Irrespective of whether either interval includes 0 and ∞ .

Calculus

The tools of calculus can be used to analyse functions of $\widehat{\mathbb{R}}$. The definitions are motivated by the topology of this space.

Neighbourhoods

Let $x \in R, A \subseteq R$

- *A* is a neighbourhood of *x*, if and only if *A* contains an open interval *B* and $x \in B$.
- *A* is a right-sided neighbourhood of x, if and only if there is $y \in \mathrm{R}\ \mathrm{y} > \mathrm{x}$ such that A contains $[x, y)$.

- *A* is a left-sided neighbourhood of x, if and only if there is $y \in \mathrm{R}\, y < x$ such that A contains $[y, x)$.

- *A* is a (right-sided, left-sided) punctured neighbourhood of *x*, if and only if there is $B \subseteq R$ such that *B* is a (right-sided, left-sided) neighbourhood of x, and $A = B \setminus \{x\}$.

Basic definitions of limits

Let $f: \hat{\mathbb{R}} \to \hat{\mathbb{R}}, p \in \hat{\mathbb{R}}$

The limit of *f(x)* as *x* approaches *p* is *L*, denoted

$$\lim_{x \to p} f(x) = L$$

if and only if for every neighbourhood A of L. there is a right-sided (left-sided) punctured neighbourhood B of ρ, such that $x \epsilon B$ implies $f(x) \epsilon A$.

The one-sided limit of *f(x)* as *x* approaches *p* from the right (left) is *L*, denoted

$$\lim_{x \to p} f(x) = L \left(\lim_{x \to p} f(x) = L \right)$$

if and only if for every neighbourhood *A* of *L*, there is a right-sided (left-sided) punctured neighbourhood *B* of *p*, such that $x \epsilon B$ implies $f(x) \epsilon A$.

It can be shown that $\lim_{x \to p} f(x) = L$ if and only if both $\lim_{x \to p^+} f(x) = L$ and $\lim_{x \to p^-} f(x) = L$.

Comparison with limits in ℝ

The definitions given above can be compared with the usual definitions of limits of real functions. In the following statements, $pL \epsilon \mathbb{R}$, the first limit is as defined above, and the second limit is in the usual sense:

- $\lim_{x \to p} f(x) = L$ is equivalent to $\lim_{x \to p} f(x) = L$
- $\lim_{x \to \infty^+} f(x) = L$ is equivalent to $\lim_{x \to -\infty} f(x) = L$
- $\lim_{x \to \infty^-} f(x) = L$ is equivalent to $\lim_{x \to +\infty} f(x) = L$
- $\lim_{x \to p} f(x) = \infty$ is equivalent to $\lim_{x \to p} |f(x)| = +\infty$
- $\lim_{x \to \infty^+} f(x) = \infty$ is equivalent to $\lim_{x \to -\infty} |f(x)| = +\infty$
- $\lim_{x \to \infty^-} f(x) = \infty$ is equivalent to $\lim_{x \to +\infty} |f(x)| = +\infty$

Extended definition of limits

Let $A \subseteq \hat{\mathbb{R}}$ Then ρ is a limit point of A if every neighbourhood of ρ includes a point $\mathcal{Y} \epsilon A$ such that y $\neq x$

Let$f: \hat{\mathbb{R}} \to \hat{\mathbb{R}}, A \subseteq \widehat{\mathbb{R}}, L\epsilon\widehat{\mathbb{R}}, p\epsilon\widehat{\mathbb{R}} \cdot \rho$ a limit point of A. The of $f(x)$ as (x) approaches ρ through A is L, if and only if for every neighbourhood B of L, there is a puntured neighbourhood C of ρ, such that $x \epsilon A \cap C$ implies $f(x)\epsilon$B.

This corresponds to the regular topological definition of continuity, applied to the subspace topology on $A \cup \{p\}$. And the restriction of f to $A \cup \{p\}$

Continuity

Let

$$f: \hat{\mathbb{R}} \to \hat{\mathbb{R}}, \qquad p \in \hat{\mathbb{R}}.$$

f is continuous at ρ if and only, if is defined at ρ and

$$\lim_{x \to p} f(x) = (\rho).$$

Let

$$f: \hat{\mathbb{R}} \to \hat{\mathbb{R}}, \qquad A \subseteq \hat{\mathbb{R}}.$$

f is continuous in A if and only if for every p ϵA. f is defined at ρ and the limit f (x) as x approaches ρ through A is $f(\rho)$. An interesting feature is that every rational function P(x)/Q(x), where P(x) and Q(x) have no common factor is continuous in $\hat{\mathbb{R}}$ Also. If tan is extended so that

$$tan\left(\frac{\pi}{2} + n\pi\right) = \infty \text{ for n } \epsilon\ \mathbb{Z},$$

then tan is continuous in ℝ. However, many elementary functions, such as trigonometric and exponential functions are discontinuous at ∞ for example, sin is continuous in ℝ but discontinues at ∞.

Thus 1/x is continuous on $\hat{\mathbb{R}}$ but not on the affinity extended real number system $\overline{\mathrm{R}}$. Conversely, the function arctan can be extended continuously on $\overline{\mathrm{R}}$, but not on $\hat{\mathbb{R}}$

Check Your Progress 1

Q. 1. What is the Definition of Limit ?

………………………………………………………………………..

………………………………………………………………………..

Q. 2 what is arithmetic operation in Real Analysis?

…………………………………………………………………………

………………………………………………………………………..

As a projective range

When the real projective line is considered in the context of the real projective plane, then the consequences of Desargues' theorem are implicit. In particular, the construction of the projective harmonic conjugate relation between points is part of the structure of the real projective line. For instance, given any pair of points, the point at infinity is the projective harmonic conjugate of their midpoint.

As projectivities preserve the harmonic relation, they form the automorphisms of the real projective line. The projectivities are described algebraically as homographies, since the real numbers form a ring, according to the general construction of a projective line over a ring. Collectively they form the group PGL(2,R).

The projectivities which are their own inverses are called involutions. A **hyperbolic involution** has two fixed points. Two of these correspond to elementary, arithmetic operations on the real projective line: negation and reciprocation. Indeed, 0 and ∞ are fixed under negation, while 1 and −1 are fixed under reciprocation.

3.5 REAL PROJECTIVE PLANE

In mathematics, the real projective plane is an example of a compact non-orientable two-dimensional manifold; in other words, a one-sided surface. It cannot be embedded in standard three-dimensional space without intersecting itself. It has basic applications to geometry, since the common construction of the real projective plane is as the space of lines in R^3 passing through the origin.

The plane is also often described topologically, in terms of a construction based on the Möbius strip: if one could glue the (single) edge of the

Möbius strip to itself in the correct direction, one would obtain the projective plane. (This cannot be done in three-dimensional space without the surface intersecting itself.) Equivalently, gluing a disk along the boundary of the Möbius strip gives the projective plane. Topologically, it has Euler characteristic 1, hence a demigenus (non-orientable genus, Euler genus) of 1.

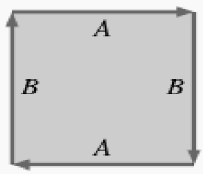
Since the Möbius strip, in turn, can be constructed from a square by gluing two of its sides together, the real projective plane can thus be represented as a unit square (that is, $[0,1] \times [0,1]$) with its sides identified by the following equivalence relations:

$(0, y) \sim (1, 1 - y)$ for $0 \leq y \leq 1$

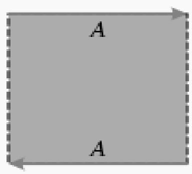
and

$(x, 0) \sim (1 - x, 1)$ for $0 \leq x \leq 1$,

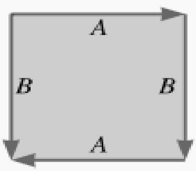
as in the shown in diagram below.



The fundamental polygon of the projective plane.



The Möbius strip with a single edge, can be closed into a projective plane by gluing opposite open edges together.



In comparison the Klein bottle is a Möbius strip closed into a cylinder.

Examples

Projective geometry is not necessarily concerned with curvature and the real projective plane may be twisted up and placed in the Euclidean plane or 3-space in many different ways. Some of the more important examples are described below.

The projective plane cannot be embedded (that is without intersection) in three-dimensional Euclidean space. The proof that the projective plane does not embed in three-dimensional Euclidean space goes like this: Assuming that it does embed, it would bound a compact region in three-dimensional Euclidean space by the generalized Jordan curve theorem. The outward-pointing unit normal vector field would then give an orientation of the boundary manifold, but the boundary manifold would be the projective plane, which is not orientable. This is a contradiction, and so our assumption that it does embed must have been false.

The projective sphere

Consider a sphere, and let the great circles of the sphere be "lines", and let pairs of antipodal points be "points". It is easy to check that this system obeys the axioms required of a projective plane:

- any pair of distinct great circles meet at a pair of antipodal points; and
- any two distinct pairs of antipodal points lie on a single great circle.

If we identify each point on the sphere with its antipodal point, then we get a representation of the real projective plane in which the "points" of the projective plane really are points. This means that the projective plane is the quotient space of the sphere obtained by partitioning the sphere into equivalence classes under the equivalence relation ~, where $x \sim y$ if $y = -x$. This quotient space of the sphere is homeomorphic with the collection of all lines passing through the origin in $\mathbf{R}^3$.

The quotient map from the sphere onto the real projective plane is in fact a two sheeted (i.e. two-to-one) covering map. It follows that the fundamental group of the real projective plane is the cyclic group of order 2, i.e. integers modulo 2. One can take the loop *AB* from the figure above to be the generator.

The projective hemisphere

Because the sphere covers the real projective plane twice, the plane may be represented as a closed hemisphere around whose rim opposite points are similarly identified.

Boy's surface – an immersion

The projective plane can be immersed (local neighbourhoods of the source space do not have self-intersections) in 3-space. Boy's surface is an example of an immersion.

Polyhedral examples must have at least nine faces.

3.5.1 Roman Surface

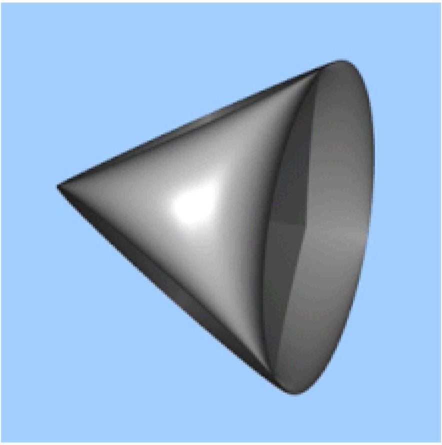
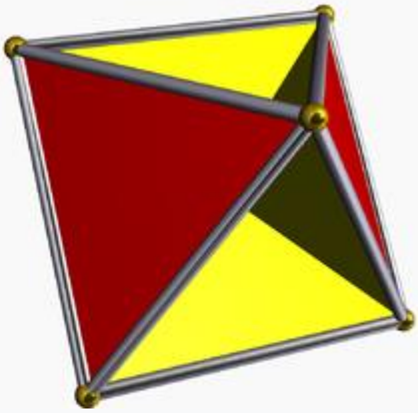


Figure of the Roman Surface

Steiner's Roman surface is a more degenerate map of the projective plane into 3-space, containing a cross-cap.



A polyhedral representation is the tetrahemihexahedron, which has the same general form as Steiner's Roman Surface, shown here.

Hemi polyhedra

Looking in the opposite direction, certain abstract regular polytopes – hemi-cube, hemi-dodecahedron, and hemi-icosahedron – can be constructed as regular figures in the *projective plane;* see also projective polyhedra.

Planar projections

Various planar (flat) projections or mappings of the projective plane have been described. In 1874 Klein described the mapping

$$k(x, y) = (1 + x^2 + y^2)^{\frac{1}{2}} \begin{pmatrix} x \\ y \end{pmatrix}$$

Central projection of the projective hemisphere onto a plane yields the usual infinite projective plane, described below.

Cross-capped disk

A closed surface is obtained by gluing a disk to a cross-cap. This surface can be represented parametrically by the following equations:

$$X(u, v) = r\,(1 + \cos v)\,\cos u,$$
$$Y(u, v) = r\,(1 + \cos v)\,\sin u,$$
$$Z(u, v) = -\tanh(u - \pi)\,r\,\sin v.$$

where both u and v range from 0 to 2π. These equations are similar to those of a torus. Figure shows a closed cross-capped disk.

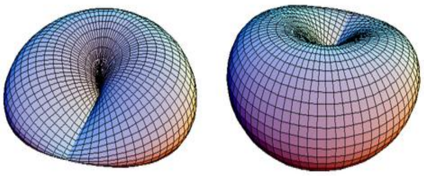


Figure. Two views of a cross-capped disk.

A cross-capped disk has a plane of symmetry which passes through its line segment of double points. In Figure 1 the cross-capped disk is seen from above its plane of symmetry $z = 0$, but it would look the same if seen from below.

A cross-capped disk can be sliced open along its plane of symmetry, while making sure not to cut along any of its double points. The result is shown in Figure

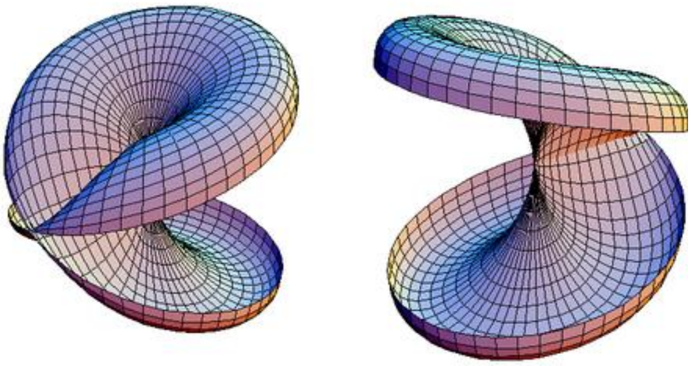


Figure. Two views of a cross-capped disk which has been sliced open.

Once this exception is made, it will be seen that the sliced cross-capped disk is homeomorphic to a self-intersecting disk, as shown in Figure .

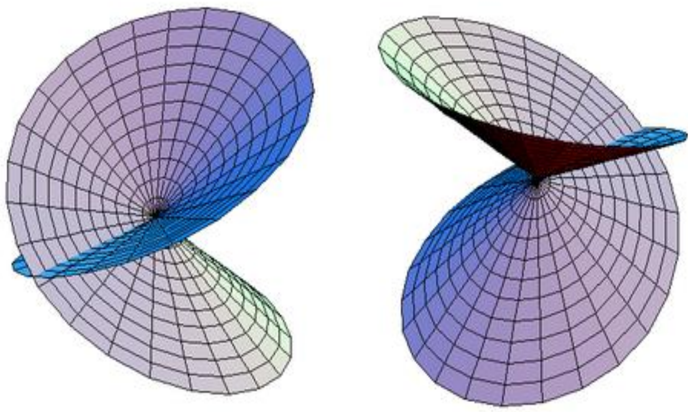


Figure. Two alternative views of a self-intersecting disk.

The self-intersecting disk is homeomorphic to an ordinary disk. The parametric equations of the self-intersecting disk are:

X(u,v)=rv cos2u,

Y(u,v)=rv sin2u

Z(u,v)=rvcosu.

where u ranges from 0 to 2π and v ranges from 0 to 1.

Projecting the self-intersecting disk onto the plane of symmetry ($z = 0$ in the parametrization given earlier) which passes only through the double points, the result is an ordinary disk which repeats itself (doubles up on itself).

The plane $z = 0$ cuts the self-intersecting disk into a pair of disks which are mirror reflections of each other. The disks have centers at the origin.

Now consider the rims of the disks (with $v = 1$). The points on the rim of the self-intersecting disk come in pairs which are reflections of each other with respect to the plane $z = 0$.

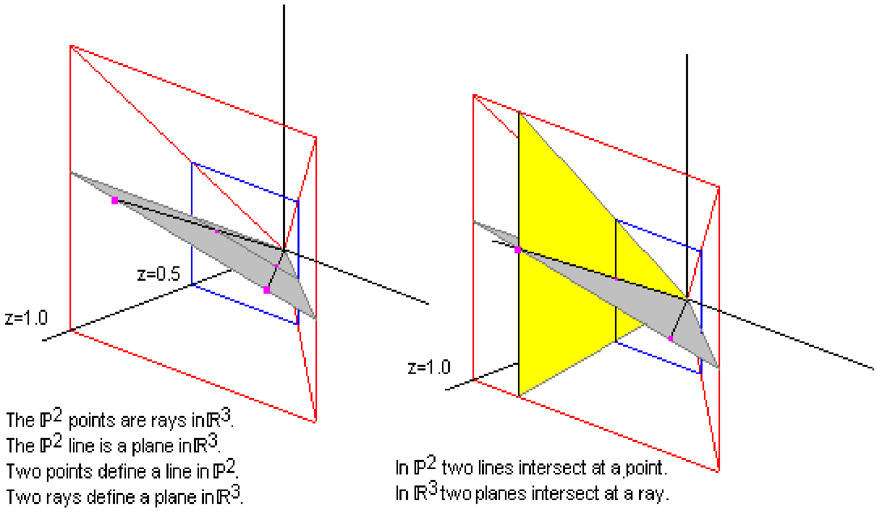
A cross-capped disk is formed by identifying these pairs of points, making them equivalent to each other. This means that a point with parameters (u,1) and coordinates (rcos2u,rsin2u,rcosu) is identified with the point ($u + \pi$,1) whose coordinates are (rcos2u,rsin2u,-rcosu) But this means that pairs of opposite points on the rim of the (equivalent) ordinary disk are identified with each other; this is how a real projective plane is formed out of a disk. Therefore the surface shown in Figure 1 (cross-cap with disk) is topologically equivalent to the real projective plane RP^2.

Homogeneous coordinates

The points in the plane can be represented by homogeneous coordinates. A point has homogeneous coordinates $[x : y : z]$, where the coordinates $[x : y : z]$ and $[tx : ty : tz]$ are considered to represent the same point, for all nonzero values of t. The points with coordinates $[x : y : 1]$ are the usual real plane, called the **finite part** of the projective plane, and points with coordinates $[x : y : 0]$, called **points at infinity** or **ideal points**, constitute a line called the **line at infinity**. (The homogeneous coordinates $[0 : 0 : 0]$ do not represent any point.)

The lines in the plane can also be represented by homogeneous coordinates. A projective line corresponding to the plane $ax + by + cz = 0$ in $\mathbf{R}^3$ has the homogeneous coordinates $(a : b : c)$. Thus, these coordinates have the equivalence relation $(a : b : c) = (da : db : dc)$ for all nonzero values of d. Hence a different equation of the same line $dax + dby + dcz = 0$ gives the same homogeneous coordinates. A point $[x : y : z]$ lies on a line $(a : b : c)$ if $ax + by + cz = 0$. Therefore, lines with coordinates $(a : b : c)$ where a, b are not both 0 correspond to the lines in the usual real plane, because they contain points that are not at infinity. The line with coordinates $(0 : 0 : 1)$ is the line at infinity, since the only points on it are those with $z = 0$.

3.5.2 Points, Lines, And Planes



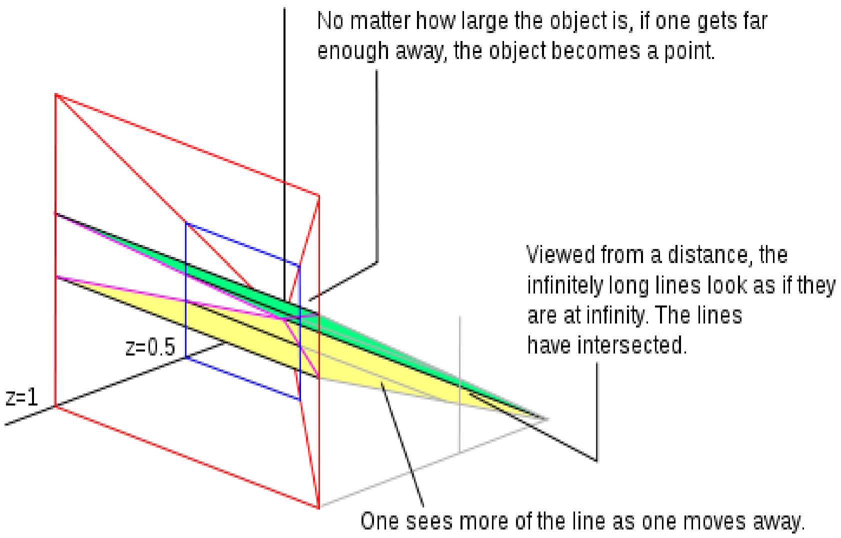
A line in $\mathbf{P}^2$ can be represented by the equation $ax + by + cz = 0$. If we treat a, b, and c as the column vector $\boldsymbol{\ell}$ and x, y, z as the column vector $\mathbf{x}$ then the equation above can be written in matrix form as:

$\mathbf{x}^T\boldsymbol{\ell} = 0$ or $\boldsymbol{\ell}^T\mathbf{x} = 0$.

Using vector notation we may instead write $\mathbf{x} \cdot \boldsymbol{\ell} = 0$ or $\boldsymbol{\ell} \cdot \mathbf{x} = 0$.

The equation $k(\mathbf{x}^T\boldsymbol{\ell}) = 0$ (which k is a non-zero scalar) sweeps out a plane that goes through zero in $\mathbf{R}^3$ and $k(x)$ sweeps out a line, again going through zero. The plane and line are linear subspaces in $\mathbf{R}^3$, which always go through zero.

Ideal points

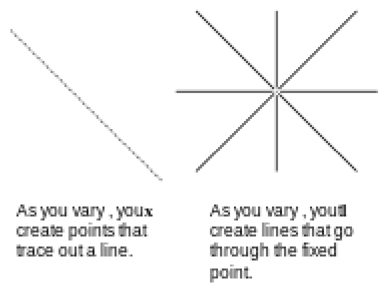


In $\mathbf{P}^2$ the equation of a line is $ax + by + cz = 0$ and this equation can represent a line on any plane parallel to the x, y plane by multiplying the equation by k.

If $z = 1$ we have a normalized homogeneous coordinate. All points that have $z = 1$ create a plane. Let's pretend we are looking at that plane (from a position further out along the z axis and looking back towards the origin) and there are two parallel lines drawn on the plane. From where we are standing (given our visual capabilities) we can see only so much of the plane, which we represent as the area outlined in red in the diagram. If we walk away from the plane along the z axis, (still looking backwards towards the origin), we can see more of the plane. In our field of view original points have moved. We can reflect this movement by dividing the homogeneous coordinate by a constant. In the adjacent image we have divided by 2 so the z value now becomes 0.5. If we walk far enough away what we are looking at becomes a point in the distance. As we walk away we see more and more of the parallel lines. The lines will meet at a line at infinity (a line that goes through zero on the plane at $z = 0$). Lines on the plane when $z = 0$ are ideal points. The plane at $z = 0$ is the line at infinity.

The homogeneous point (0, 0, 0) is where all the real points go when you're looking at the plane from an infinite distance, a line on the $z = 0$ plane is where parallel lines intersect.

Duality



In the equation $\mathbf{x}^T\boldsymbol{\ell} = 0$ there are two column vectors. You can keep either constant and vary the other. If we keep the point $\mathbf{x}$ constant and vary the coefficients $\boldsymbol{\ell}$ we create new lines that go through the point. If we keep the coefficients constant and vary the points that satisfy the equation we create a line. We look upon $\mathbf{x}$ as a point, because the axes we are using are x, y, and z. If we instead plotted the coefficients using axis marked a, b, c points would become lines and lines would become points. If you prove something with the data plotted on axis marked x, y, and z the same argument can be used for the data plotted on axis marked a, b, and c. That is duality.

Lines joining points and intersection of lines (using duality)

The equation $\mathbf{x}^T\boldsymbol{\ell} = 0$ calculates the inner product of two column vectors. The inner product of two vectors is zero if the vectors are orthogonal. In $\mathbf{P}^2$, the line between the points $\mathbf{x}_1$and $\mathbf{x}_2$ may be represented as a column vector $\boldsymbol{\ell}$ that satisfies the equations $\mathbf{x}_1{}^T\boldsymbol{\ell} = 0$ and $\mathbf{x}_2{}^T\boldsymbol{\ell} = 0$, or in other words a column vector $\boldsymbol{\ell}$ that is orthogonal to $\mathbf{x}_1$ and $\mathbf{x}_2$. The cross product will find such a vector: the line joining two points has homogeneous coordinates given by the equation $\mathbf{x}_1 \times \mathbf{x}_2$. The intersection of two lines may be found in the same way, using duality, as the cross product of the vectors representing the lines, $\boldsymbol{\ell}_1 \times \boldsymbol{\ell}_2$.

Embedding into 4-dimensional space

The projective plane embeds into 4-dimensional Euclidean space. The real projective plane $\mathbf{P}^2(\mathbf{R})$ is the quotient of the two-sphere

$\mathbf{S}^2 = \{(x, y, z) \in \mathbf{R}^3 : x^2+y^2+z^2 = 1\}$

by the antipodal relation $(x, y, z) \sim (-x, -y, -z)$. Consider the function $\mathbf{R}^3 \to \mathbf{R}^4$ given by $(x, y, z) \mapsto (xy, xz, y^2-z^2, 2yz)$. This map restricts to a map whose domain is $\mathbf{S}^2$ and, since each component is a homogeneous polynomial of even degree, it takes the same values in $\mathbf{R}^4$ on each of any two antipodal points on $\mathbf{S}^2$. This yields a map $\mathbf{P}^2(\mathbf{R}) \to \mathbf{R}^4$. Moreover, this map is an embedding. Notice that this embedding admits a projection into $\mathbf{R}^3$ which is the Roman surface.

Higher non-orientable surfaces

By gluing together projective planes successively we get non-orientable surfaces of higher demigenus. The gluing process consists of cutting out a little disk from each surface and identifying (*gluing*) their boundary circles. Gluing two projective planes creates the Klein bottle.

The article on the fundamental polygon describes the higher non-orientable surfaces.

In mathematics, a **projective line** is, roughly speaking, the extension of a usual line by a point called a *point at infinity*. The statement and the proof of many theorems of geometry are simplified by the resultant elimination of special cases; for example, two distinct projective lines in a projective plane meet in exactly one point (there is no "parallel" case).

There are many equivalent ways to formally define a projective line; one of the most common is to define a projective line over a field *K*, commonly denoted $\mathbf{P}^1(K)$, as the set of one-dimensional subspaces of a two-dimensional *K*-vector space. This definition is a special instance of the general definition of a projective space.

Homogeneous coordinates

An arbitrary point in the projective line $\mathbf{P}^1(K)$ may be represented by an equivalence class of *homogeneous coordinates*, which take the form of a pair

$$[x_1 : x_2]$$

of elements of *K* that are not both zero. Two such pairs are equivalent if they differ by an overall nonzero factor *λ*:

$$[x_1 : x_2] \sim [\lambda x_1 : \lambda x_2].$$

Line extended by a point at infinity

The projective line may be identified with the line *K* extended by a point at infinity. More precisely, the line *K* may be identified with the subset of $\mathbf{P}^1(K)$ given by

$$\left\{[x : 1] \in \mathbf{P}^1(K) \mid x \in K\right\}.$$

This subset covers all points in $\mathbf{P}^1(K)$ except one, which is called the *point at infinity*:

$$\infty = [1 : 0].$$

This allows to extend the arithmetic on *K* to $\mathbf{P}^1(K)$ by the formulas

$$\frac{1}{0} = \infty, \qquad \frac{1}{\infty} = 0,$$

Translating this arithmetic in terms of homogeneous coordinates gives, when [0 : 0] does not occur:

Real projective line

The projective line over the real numbers is called the **real projective line**. It may also be thought of as the line *K* together with an idealised *point at infinity* ∞; the point connects to both ends of *K* creating a closed loop or topological circle.

An example is obtained by projecting points in $\mathbf{R}^2$ onto the unit circle and then identifying diametrically opposite points. In terms of group theory we can take the quotient by the subgroup $\{1, -1\}$.

Compare the extended real number line, which distinguishes ∞ and $-\infty$.

Complex projective line: the Riemann sphere

Adding a point at infinity to the complex plane results in a space that is topologically a sphere. Hence the complex projective line is also known as the **Riemann sphere** (or sometimes the *Gauss sphere*). It is in constant use in complex analysis, algebraic geometry and complex manifold theory, as the simplest example of a compact Riemann surface.

For a finite field

The projective line over a finite field F_q of *q* elements has $q + 1$ points. In all other respects it is no different from projective lines defined over other types of fields. In the terms of homogeneous coordinates [*x* : *y*], *q* of these points have the form:

[*a* : 1] for each *a* in F_q,

and the remaining point *at infinity* may be represented as [1 : 0].

Symmetry group

Quite generally, the group of homographies with coefficients in *K* acts on the projective line $\mathbf{P}^1(K)$. This group action is transitive, so that $\mathbf{P}^1(K)$ is a homogeneous space for the group, often written $\mathrm{PGL}_2(K)$ to emphasise the projective nature of these transformations. *Transitivity* says that there exists a homography that will transform any point *Q* to any other point *R*.

The *point at infinity* on $\mathbf{P}^1(K)$ is therefore an *artifact* of choice of coordinates: homogeneous coordinates.

express a one-dimensional subspace by a single non-zero point (X, Y) lying in it, but the symmetries of the projective line can move the point $\infty = [1 : 0]$ to any other, and it is in no way distinguished.

Much more is true, in that some transformation can take any given distinct points Q_i for $i = 1, 2, 3$ to any other 3-tuple R_i of distinct points (*triple transitivity*). This amount of specification 'uses up' the three dimensions of $PGL_2(K)$; in other words, the group action is sharply 3-transitive. The computational aspect of this is the cross-ratio. Indeed, a generalized converse is true: a sharply 3-transitive group action is always (isomorphic to) a generalized form of a $PGL_2(K)$ action on a projective line, replacing "field" by "KT-field" (generalizing the inverse to a weaker kind of involution), and "PGL" by a corresponding generalization of projective linear maps.[1]

As algebraic curve

The projective line is a fundamental example of an algebraic curve. From the point of view of algebraic geometry, $\mathbf{P}^1(K)$ is a non-singular curve of genus 0. If K is algebraically closed, it is the unique such curve over K, up to rational equivalence. In general a (non-singular) curve of genus 0 is rationally equivalent over K to a conic C, which is itself birationally equivalent to projective line if and only if C has a point defined over K; geometrically such a point P can be used as origin to make explicit the birational equivalence..

The function field of the projective line is the field $K(T)$ of rational functions over K, in a single indeterminate T. The field automorphisms of $K(T)$ over K are precisely the group $PGL_2(K)$ discussed above.

Any function field $K(V)$ of an algebraic variety V over K, other than a single point, has a subfield isomorphic with $K(T)$. From the point of view of birational geometry, this means that there will be a rational map from V to $\mathbf{P}^1(K)$, that is not constant. The image will omit only finitely many points of $\mathbf{P}^1(K)$, and the inverse image of a typical point P will be of dimension $\dim V - 1$. This is the beginning of methods in algebraic geometry that are inductive on dimension. The rational maps play a role analogous to the meromorphic functions of complex analysis, and indeed in the case of compact Riemann surfaces the two concepts coincide.

If *V* is now taken to be of dimension 1, we get a picture of a typical algebraic curve *C* presented 'over' $\mathbf{P}^1(K)$. Assuming *C* is non-singular (which is no loss of generality starting with $K(C)$), it can be shown that such a rational map from *C* to $\mathbf{P}^1(K)$ will in fact be everywhere defined. (That is not the case if there are singularities, since for example a *double point*where a curve *crosses itself* may give an indeterminate result after a rational map.) This gives a picture in which the main geometric feature is ramification.

Many curves, for example hyperelliptic curves, may be presented abstractly, as ramified covers of the projective line. According to the Riemann–Hurwitz formula, the genus then depends only on the type of ramification.

A **rational curve** is a curve that is birationally equivalent to a projective line (see rational variety); its genus is 0. A rational normal curve in projective space P^n is a rational curve that lies in no proper linear subspace; it is known that there is only one example (up to projective equivalence),[2] given parametrically in homogeneous coordinates as

$[1 : t : t^2 : ... : t^n]$.

3.6 LETS SUM UP

Various ideas from real analysis can be generalized from the real line to broader or more abstract contexts. These generalizations link real analysis to other disciplines and sub disciplines, in many cases playing an important role in their development as distinct areas of mathematics. For instance, generalization of ideas like continuous functions and compactness from real analysis to metric spaces and topological spaces connects real analysis to the field of general topology, while generalization of finite-dimensional Euclidean spaces to infinite-dimensional analogs led to the study of Banach spaces, and Hilbert spaces as topics of importance in functional analysis. Georg Cantor's investigation of sets and sequence of real numbers, mappings between them, and the foundational issues of real analysis gave birth to naive set theory. The study of issues of convergence for sequences of functions eventually gave rise to Fourier analysis as a sub discipline of mathematical analysis. Investigation of the consequences of generalizing differentiability from functions of a real variable to ones of a complex variable gave rise to the concept of holomorphic functions and the inception of complex analysis as another distinct sub discipline of analysis. On the other hand, the generalization of integration from the Riemann sense to that of Lebesgue led to the formulation of the concept of abstract measure spaces, a fundamental concept in measure theory. Finally, the generalization of integration from the real line to curves and surfaces in higher dimensional space brought about the study of vector calculus, whose further generalization and formalization played an important role in the evolution of the concepts of differential forms and smooth (differentiable) manifolds in differential geometry and other closely related areas of geometry and topology.

3.7 KEYWORD

ROMAN PLANE

PROJECTIVE PLANE

EMBEDDING

MEROMORPHIC

HAMYPOLYHEDRA

CAPP DISK

MEASURE

EXTENSIONS

MOTIVATION

CAUCHY SEQUENCE

RIEMAAN SEQUENCE

3.8 QUESTIONS FOR REVIEW

Q.1 Describe the homogeneous coordinates?

Q.2 Discuss Roman Surface?

Q.3 Discuss real projective plane in three dimensions?

Q. 4 What is arithmetic operations in real analysis?

Q. 5 What is complex projective plane?

3.9 SUGGESTESTED READINGS & REFFERENCE

1. *Tao, Terence (2003). "Lecture notes for MATH 131AH" (PDF). Course Website for MATH 131AH, Department of Mathematics, UCLA.*
2. ^ Some authors (e.g., Rudin 1976) use braces instead and write However, this notation conflicts with the usual notation for a set, which, in contrast to a sequence, disregards the order and the multiplicity of its elements.
3. ^ *Stewart, James (2008). Calculus: Early Transcendentals (6th ed.). Brooks/Cole. ISBN 0-495-01166-5.*
4. ^ Royden 1988, Sect. 5.4, page 108; Nielsen 1997, Definition 15.6 on page 251; Athreya & Lahiri 2006, Definitions 4.4.1, 4.4.2 on pages 128,129. The interval *I* is assumed to be bounded and closed in the former two books but not the latter book.

3.10 ANSWERS TO CHECK YOUR PROGRESS

1. Hint please check Calculus section

2. Hint Please check section 3.3.2 Arithmetic operations

UNIT 4 ALGEBRAIC OPERATIONS

STRUCTURE

4.1 THE MEANING OF ALGEBRA

The previous articles have been concerned with the arithmetic operations involving addition, subtraction, multiplication, and division. These operations have been applied to whole numbers and fractions. Algebra involves the extension of these principles to symbols that are used to represent numbers. The symbols are used to write arithmetic statements for formulas into which many sets of specific numerical values can be substituted. Therefore, algebra may be called generalized arithmetic. For example, the area of a rectangle equals the length times the width. This is represented by the algebraic expression A = lw.

Area = length x width

The above statement for the area of a rectangle is always true. To compute a numerical value for the area of a specific rectangle, the numerical values of the length, l, and the width, w, are substituted and the indicated multiplication carried out. Letters are used to represent numbers. Since these letters represent numbers, they are subject to the same rules developed for whole numbers and fractions.

4.2 SIGNED NUMBERS

Before taking up the study of algebra, it is necessary to become familiar with the concept of signed numbers. The numbers that are used to describe the number of objects in a group, and the counting numbers, are always positive numbers. That is, they are always greater than zero. However, there are many occasions when negative numbers, numbers less than zero, must be used. These numbers arise when describing measurement in a direction opposite to the positive numbers. For example, if assigning a value of +3 to a point which is 3 feet above the ground, what numbers should be assigned to a point which is 3 feet below the ground? Perhaps the most familiar example of the use of negative numbers is the measurement of temperature, where temperatures below an arbitrary reference level are assigned negative values.

Every number has a sign associated with it. The plus (+) sign indicates a positive number; the minus (-) sign indicates a negative number. When no sign is given, a plus sign is implied. The fact that the plus and minus signs are also used for the arithmetic operations of addition and subtraction should not be the cause for confusion since their meanings are equivalent.

Every number has an absolute value, regardless of its sign. The absolute value for a number is indicated by a pair of vertical lines enclosing the number. The absolute value indicates the distance from zero, without regard to direction. The number +5 is 5 units from zero, in the positive direction. The number -5 is also 5 units from zero, in the negative direction. The absolute value of each of these numbers is 5.

The absolute value of -5, written as $|-5|$, = 5.

$|-5| = 5$

The absolute value of +5, written as |+5|, is also 5.

|+5| = 5

Therefore,

|-5| = |+5| = 5

4.2.1 Operations with Signed Numbers

Performing the operations of addition, subtraction, multiplication, and division are more easily visualized if the numbers are placed on a number line. The positive numbers are greater than zero, and lie to the right of zero on the number line. The negative numbers are less than zero, and lie to the left of the zero on the number line.

-4 -1 -2 -10 +1 +2 +3 +4

The number line extends an infinite distance in each direction and therefore includes all numbers. The process of addition can be considered as counting to the right on the number line. For example when adding 1+2 the location of +1 must be found on the number line. In this example 2 units must be counted to the right, for adding. The result is +3. To illustrate further, when adding -2 and +4 the location of -2 on the number line must first be found. Counting 4 units to the right ends up at +2.

The number line is useful for illustrating the principles of addition, but it clearly would be inconvenient to use in the case of large numbers. Consequently, the following rules were developed to govern the addition process:

- To add two numbers with like signs, find the sum of their absolute values and prefix the common sign.
- To add two numbers with unlike signs, find the difference between their absolute values and prefix the sign of the number having the greater absolute value.
- To add more than two numbers, combine the positive and negative numbers separately using the rule for numbers with like signs and then combine the two results using the rule for numbers with unlike signs. A few examples are:

1. **Add -37 and -16.**

Since these numbers have like signs, add their absolute values and prefix the common sign.

|-37| = 37 37

|-16| = 16 +16

53

Answer= -53

2. **Add -37 and +16.**

Since these numbers have unlike signs, take the difference between their absolute values and prefix the sign of the number having the greater absolute value.

|-37| = 37 37

|+16| = 16 -16

21

Answer= -21

3. **Add +32, -16, -19, -12 and +14**.

First, combine the positive and negative numbers separately.

Positive Negative

32 -16

14 -19

46 -12

-47

Then combine the results: (+46) + (-47) = -1

The subtraction of signed numbers is governed by only one rule. To subtract two signed numbers, change the sign of the subtrahend and then add the two numbers as shown below.

Subtract 7 from 11. (11) - (7) = (11) + (-7) = 4

Subtract -7 from 11. (11) - (-7) = (11) + (+7) = 18

Subtract -7 from -11. (-11) - (-7) = (-11) + (+7) = -4

Subtract 7 from -11. (-11) - (7) = (-11) + (-7) = -18

In each of these examples, the subtrahend was changed, then the rules for addition were followed.

The multiplication and division of signed numbers are governed by the following rules:

- The product of two numbers with like signs is a positive number. The product of two numbers with unlike signs is a negative number. In symbols:

  (+) x (+) = (+) (+) x (-) = (-)
  (-) x (-) = (+) (-) x (+) = (-)

- The division of numbers with like signs gives a positive quotient. The division of numbers with unlike signs gives a negative quotient. In symbols:

  (+) ÷ (+) = (+) (+) ÷ (-) = (-)
  (-) ÷ (-) = (+) (-) ÷ (+) = (-)
  Examples:

1. Multiply +4 and –3 (+4) x (-3) = -12
2. Divide -24 by –6 (-24) ÷ (-6) = +4

Remember that multiplication is really a short form of addition. When multiplying +4 by -3, the number -3 is added four times. That is, (-3) + (-3) + (-3) + (-3) = -12. Also, since division is a short form of subtraction, the number -6 is subtracted from -24 four times in order of reach 0, i.e., 24 - (-6) - (-6) - (-6) - (-6) = 0. Although the process could be repeated for the multiplication and division of any signed numbers, usage of the two rules will produce equivalent results.

4.2.2 Algebraic Expressions and Terms

An algebraic term is "a combination of numbers and letters (literal numbers) linked by multiplication or division." There is no limit to the number of quantities in the term. The following are all algebraic terms:

1. 3 6. (2xz)/ y
2. 3x 7. 3abcxy2
3. 4ab 8. (5bcx2yz)/ 3a
4. (7x2)\\ y 9. 19/ abcxy
5. 11xyz 10. X2

An algebraic expression is "a sum or difference of algebraic terms." There is no limit as to the number of terms in the expression.

The following are all algebraic expressions:

1. 3 + 4ab

2. 11xyz + (2xz)/y

3. 7x2/y + 19/abcxy + y2 + (2xy)/ z

The factors of a term are each of the numbers or letters that, when combined (multiplied or dividend), produce the term. For example:

1. 5, x, and y are factors of the term 5xy.
2. 7, x, and y are factors of the term (7x2)/y
3. The numbers 3 and 5 are factors of the term 15.

Notice that numbers are often referred to as factors, as was done in example three above. Numbers may have many factors.

Prime numbers are those which have only themselves and 1 as factors.

Examples: 3, 7, 11, 13, and 17 are prime numbers. Their only factors are themselves and 1.

When considering the factors of numbers, only the factors which are whole numbers are considered. In the example above, the numbers 3.5 and 2 are factors of 7 in the sense that 3.5 x 2 = 7, but of course 3.5 is not a whole number.

The word coefficient is used for the numerical factors in an algebraic term. In the term 5xy, the number 5 is the coefficient of xy. The coefficient need not be a whole number.

As we said before, algebraic expressions may have any number of terms. These expressions have special names, which indicate the number of terms contained in them. A monomial is an algebraic expression consisting of only one term. A polynomial is an algebraic expression, which contains more than one term. Special names for polynomials are binomials, consisting of two terms, and trinomials, consisting of three terms.

Examples:

1. 3x2y and 2x are monomials.
2. 3x2y + 2x is a binomial.
3. 3x2y + 2x + z is a trinomial.

4.2.3 Addition and Subtraction of Expressions

Suppose there are two terms, 2 and 3, and their sum is desired: 2 + 3. The sum, of course, is represented by the single symbol 5. This really means that 2 units plus 3 units is equal to 5 units. If literal numbers are used instead of numerical ones, they can be added or subtracted provided that they have the same units. The literal number x added to the literal number 2x will result in the number 3x. Algebraic terms having exactly the same letter parts are called like terms, and may be added or subtracted by adding or subtracting the numerical coefficients. For example:

1. **Add 3x2, -2x2, and 7x2.**

(3x2), (-2x2), and (7x2) = 8x2

2. **Add (3x2 - 2xy + 5y2) and (x2 + y2).**

3x2 - 2xy + 5y2

+ x2 + y2

4x2 - 2xy = 6y2

Notice that the like terms are added or subtracted directly. Unlike terms cannot be combined even when they contain several of the same letters. The terms must be exactly the same. For example:

1. 3x + 2y don't equal 5xy, just as 3 apples + 2 oranges don't equal 5 apple-oranges.
2. 3x3y3z4 + 5x2y3z4 cannot be combined further.

In the second example above, the terms are very similar but one has a factor of x3 and the other a factor of x2. This prevents them from being combined.

4.2.4 Use of Parentheses

Whenever algebraic expressions are written out horizontally, there will always be confusion regarding just what operations are to be performed. Consider, for example, the expression 3 + x ÷ 2 + 5. This can have many different interpretations. It could mean add 3 and x, and divide the sum by 7, or divide x by 7 and add it to 3, or add 3 and x, divide by 2, and add 5, etc. Parentheses or brackets are used to group the quantities in the exact order in which the arithmetic operations are to be performed. In the example above, the expression to mean add 3 and x, and divide the sum by 7, is written:

(3 + x) ÷ (2 + 5)

This indicates that the entire quantity (3 + x) is to be divided by the entire quantity (2 + 5). Other examples are:

1. (x + Y) ÷ 2 is used to indicate the entire quantity x + y is to be divided by 2.
2. 4(x + 2) means 4 times the quantity (x + 2). It also equals 4x + 8.

The following rules govern the use of parentheses:

- If a quantity within parentheses is preceded by a plus sign (+), this plus sign and the parentheses may be omitted without changing the sign of any term within the parentheses.
- If a quantity within parentheses is preceded by a minus sign (-), this minus sign and the parentheses may be omitted provided that the sign of each term within the parenthesis changed.
- If a quantity within parentheses is preceded or followed by another quantity without an intervening sign, multiplication is indicated and the parentheses can be omitted provided that each term within the parentheses is multiplied by the quantity immediately preceding or following the parentheses.

When the parentheses are used in algebraic expressions, they are used in closed pairs. Such expressions are evaluated by working from the innermost to the outermost closed pair, as shown in the following examples:

1. Evaluate (3xy) + (3x2 - 5xy) - (4xy + 3z).

(3xy) + (3x2 - 5xy) - (4xy + 3z)

Rule 1 Rule 2

= 3xy + 3x2 - 5xy - 4xy -3z

= 3x2 - 3z - 6xy = 3(x2 - z - 2xy)

2. Evaluate 3x - 2(5y -(4x +2))

3x - 2(5y - (4y +2))

= 3x -2(5y - 4x -2)

Rule 3

= 3x -10y + 8x =4

= 11x - 10y + 4

Note the points in the examples where the rules have been applied.

Check your Progress -1

1. What is the meaning of Algebra?

…………………………………………………………………………

………………………………………………………………………..

2. What is the Signed Number?

…………………………………………………………………………

………………………………………………………………………..

4.3 ALGEBRAIC OPERATIONS II

Multiplication and Division of Algebraic Expressions

The last section covered algebraic terms and how they can be combined by addition or subtraction only when they are exactly alike, except for their numerical coefficients. However, unlike terms can be multiplied or divided directly as shown below.

1. Multiply 4a by 3b.(4a)(3b) = 12ab
2. Multiply xyz by 2ab.(xyz)(2ab) = 2abxyz
3. Divide 4a by 2b.(4a/2b) = (2a/b)

To multiply or divide terms that include the same literal number (letter), the rules which apply to this type of operation must be introduced. This subject is treated in much greater detail elsewhere, but for now it is sufficient to state The following rule:

For factors which are the same base raised to a power (exponent), the factors are multiplied by adding the exponents and are divided by subtracting the exponents. The following example demonstrates this rule:

1. Multiply a3 by a4. a3 = a . a . a a4 = a . a . a . a a3 x a4 = (a . a . a)(a . a . a . a)

= (a . a . a . a . a . a . a)

= a7

In applying this rule to a number that does not have an exponent associated with it, that number is treated as having an exponent of +1. For example:

1. Multiply x2 and x3.

(x2)(x3) = x2+3 = x5

2. Divide x5 by x2.

(x5) ÷ (x2) = x5-2 = x3

3. Multiply (x2yz) and (x5y2).

(x2yz)(x5y2) = x2+5y1+2z = x7y3z

Multiplication of monomials follows the following three steps:

Step 1. Determine the signs of the product by the rule for the multiplication of signed numbers.

Step 2. Multiply the numerical coefficients of the factors to get the numerical coefficient of the product.

Step 3. Multiply the literal parts by writing down all literal factors and adding the exponents of like literal factors.

Examples:

1. (3abx2) (-4bcx3)

-(3) (4) ab1+1cx2+3

= -12ab2cx5

2. (7x2yz3) (4xy2z3)

= (7) (4) x2+1y1+2z3+3

= 28x3y3z6

The division of monomials is done in exactly the same way, except the rules for division rather than multiplication is used.

Step 1. Determine the sign of the quotient by the rule for the division of signs.

Step 2. Divide the numerical coefficients to get the numerical coefficient of the quotient.

Step 3. Divide the literal parts by writing down all the literal factors of the dividend and subtracting the exponents of like factors in the divisor.

Examples:

1. Divide 4x2yz by 2xab.

((4x2yz)/ (2xab)) = 4/2 (((x2 - 1)(y1)(z1))/ (ab)) = 2 ((xyz)/ (ab))

2. Divide (-36a2bx3y4z) by 4abxy4.

$$\frac{-36a^2bx^3y^4z}{4abxy^4} = -\left(\frac{36}{4}\right)(a^{2-1})(b^{1-1})(x^{3-1})(y^{4-4})z^1 = -9ax^2z$$

The multiplication and division of monomials involves a straight-forward multiplication and division of terms. The multiplication of polynomials is only slightly more complicated. To multiply polynomials, multiply each term of one polynomial by each term of the other, and then combine any like terms. As an example, multiply the two numbers 6 and 8. The answer is 48. If instead, the number 6 is represented as (4 + 2) and the number 8 as (5 + 3), the product can also be written as (4 + 2)(5 + 3), a product of two polynomials. Each term of one polynomial is multiplied by each term of the other and combined.

(4 + 2)(5 + 3) = (4 x 5) + (4 x 3) + (2 x 5) + (2 x 3)

= 20 + 12 + 10 + 6

= 48

The multiplication process is done in exactly the same way for algebraic polynomials.

Example: Multiply (2x2 + x - 3) and (6x2 - 2x - 5).

(2x2 + x - 3)(6x2 - 2x - 5)

= (2x2)(6x2) - (2x2)(2x) - (2x2)(5) + x(6x2) - (x)(2x) - (x)(5)

- 3(6x2) + 3(2x) + (3)(5)

= 12x4 - 4x3 - 10x2 + 6x3 - 2x2 - 5x - 18x2 + 6x + 15

Combining like terms:

= 12x4 + 2x3 - 30x2 + x + 15

Factoring

Factoring is the reverse of multiplication. For example, the numbers 2 and 3 are factors of the number 6, since 2 x 3 = 6. In the same way, the factors of the polynomial are two (or more) other polynomials which when multiplied yield the original polynomial. In the example at the end of the last section, the two factors of the polynomial (12x4 + 2x3 - 30x2 + x + 15) are (2x2 + x - 3) and (6x2 - 2x - 5). Therefore, the process of factoring involves finding those factors as shown below.

Certain algebraic expressions can be factored by extracting a factor which is common to each term. The following rules apply:

1. Inspect the terms of polynomial to determine the greatest common factor that is contained in each term. This common factor is one factor.
2. Divide this common factor into each term of the polynomial. The quotient is the other factor.

Factor: (9x3 + 6x2 + 3x)

The greatest common factor of each term is 3x. Dividing 3x into each term yields 3x2 + 2x + 1. Therefore, the two factors are:

(3x) (3x2 + 2x + 1)

The factoring process can be checked simply by multiplying the two factors together and obtaining the original polynomial.

Certain types of binomials can be factored immediately. If the binomial is composed of the difference of two perfect squares, the two factors will be the sum and difference of the square roots of each term in the binomial. This is shown below by factoring

(81x4 - 4)

(81x4 - 4) is the difference between two squares. The square root of 81x4 is 9x2; the square root of 4 is 2. Therefore, the two factors are:

(9x2 + 2) and (9x2 - 2)

Therefore, (81x4 - 4) = (9x2 + 2)(9x2 - 2)

Note that this rule does not apply to a binomial, which is the sum of two squares, only the difference of two squares.

The most complicated polynomial usually dealt with is the trinomial, in particular, trinomials which are in the form ax2 + bx + c. The a, b, and c are positive or negative numbers. There is a straight forward procedure which will yield the factors. First, multiply a and c, to obtain the product ac. Then list all the factors of ac.

Example: Factor (4x2 - 10x -6)

Here, a = 4, c = -6.

The product of a and c is (4)(-6) = -24

The factors of -24 are:

-24, 1 24, -1

-12, 2 12, -2

-8, 3   8, -3

-6, 4 6, -4

The pair of factors whose algebraic sum is equal to b is then found. For example, in the [removed]4x2 - 10x -6), b = -10. The factors of -24 whose algebraic sum is -10 are -12, 2 since -12 + 2 = -10.

Example: The factors of (4x2 - 10x - 6) are:

$$\frac{(4x-12)(4x+2)}{4} = (x-3)(4x+2)$$

The two factors obtained can be checked by multiplication.

For another example: Find the two factors of (3x2 + 4x - 15).

1. Find factors of (3) (-15) = -45

-45, 1 45, -1

-15, 3 15, -3

-9, 5 9, -5

2. Find the two factors whose algebraic sum is +4. They are 9, -5, since 9 + (-5) = 4.

3. The factors are: $\frac{(3x+9)(3x-5)}{3} = (x+3)(3x+5)$

Every trinomial can be factored in this matter. There may be cases, however, where the numbers may not turn out to be whole numbers, but the factoring process will still be valid.

Algebraic Fractions

All operations that were developed with regard to numerical fractions apply directly to algebraic fractions. If the numerator and denominator of any fraction is multiplied or divided by the same quantity, the value of the fraction is unchanged. If the fractions are to be added or subtracted, they must have the same denominators. Finally, the fractions should be reduced to their simplest form. These rules apply to algebraic fractions because the letters simply represent numbers, and so they must obey the same rules.

Algebraic fractions can frequently be reduced by factoring both the numerator and denominator, and then dividing out any common factors. For example:

Reduce the fraction: $\frac{(x^2 - 9)}{3x^2 - 8x - 3}$

The numerator of the fraction is the difference between two squares, and so can be factored as (x + 3) (x -3).

The denominator can be factored according to the method developed for factoring trinomials. Its two factors are: (3x + 1)(x -3).

The fraction is now: $\frac{(x+3)(x-3)}{(3x+1)(x-3)}$

Since the factor (x -3) is common to both numerator and denominator, it can be divided out, leaving:

(x + 3)

(3x + 1)

Fractions are added or subtracted by finding the lowest common denominator and then combining the numerators. To demonstrate this adds the fractions:

$$\frac{x+2}{x-3} + \frac{x-1}{x+3}$$

The lowest common denominator is (x + 3)(x -3). Rewriting each fraction with the LCD as its denominator:

$$\frac{(x+2)(x+3)}{(x-3)(x+3)} + \frac{(x+1)(x-3)}{(x-3)(x+3)}$$

$$= \frac{(x+2)(x+3) + (x+1)(x-3)}{(x-3)(x+3)}$$

$$\frac{x^2 + 5x + 6 + x^2 - 2x - 3}{(x-3)(x+3)}$$

$$= \frac{2x^2 + 3x + 3}{(x-3)(x+3)} = \frac{2x^2 + 3x + 3}{(x^2 - 9)}$$

The procedure is just the same as for numerical fractions.

Changing Signs

When dealing with algebraic expressions and numerical expressions, use has been made of parentheses to group terms so that the operations of addition, subtraction, multiplication, and division are done in the proper order. When expressions are combined, the parentheses are removed, and care must be taken to properly account for the algebraic signs.

In an algebraic expression, parentheses which are preceded by a minus(-) sign may be removed by changing the sign of each term in the parentheses. For example, -(2x2 + 3x - 7) becomes -2x2 - 3x + 7. The expression is equivalent to + (-1)(2x2 + 3x -7).

Parentheses which are preceded by a plus (+) sign may be removed without changing the sign of any term in the parentheses + (2x2 + 3x - 7) would be equal to 2x2 + 3x - 7.

If the parentheses are preceded by some quantity other than -1 or +1, this indicates that the multiplication is to be performed if the parentheses are to be removed. This is really just an application of the rule for multiplying polynomials, where each term of one polynomial multiplies every term of the other. For example:

-3(2x2 + 3x -7)

= -(6x2 + 9x - 21)

= -6x2 - 9x + 21

In connection with any fraction, there are always three signs associated with it:

- the sign of the numerator
- the sign of the denominator
- the sign of the fraction itself

If any two of these three signs are changed, the value of the fraction will not be changed. The following example is used to demonstrate this.

$$-\frac{x^2-4x-7}{3x-5-x^2}$$

Use parentheses to show explicitly the three signs:

$$-\frac{+\left(x^2-4x-7\right)}{+\left(3x-5-x^2\right)}=\frac{-\left(x^2-4x-7\right)}{--\left(3x-5-x^2\right)}=+\frac{-\left(x^2 4x-7\right)}{+\left(3x-5-x^2\right)}=+\frac{+\left(x^2-4x-7\right)}{-\left(3x-5-x^2\right)}$$

All of these fractions have the same value. What is really being done is multiplying the numerator and denominator by the same quantity, in this case -1. This operation does not change the value of a fraction.

4.4 ALGEBRAIC EQUATIONS

Definition:

An equation is "a mathematical statement which says that two quantities are equal." The equality is expressed by the equal (=) sign between the two quantities. Thus, the statement $3 = 3$ is an equation. It says that the quantity to the left of the (=) sign is equal to the quantity to the right of the (=) sign. Similarly, the statement $x = 3$ is an equation. It says that the variable represented by the letter x has the value of 3.

There are two types of equations, identities and conditional equations. An identity is an equation which is always true no matter what numerical value is given to the letter which represents a variable, or unknown.

$3x + 5x = 8x$ is an identity since it is true for all values of x.

The equations which are dealt with most of the time are conditional equations. These are equations which are true only for some particular value or values of the unknown.

$3x + 5 = 8$ is a conditional equation since it is true only for the value of $x = 1$.

One of the fundamental purposes of algebra is to determine the value of the unknown which makes an equation true; that is, which "satisfies" the equation. In the example above, it can be determined that the value $x = 1$ satisfies the equation by replacing the letter x by the number 1 and performing the indicated arithmetical operations. Algebra provides the mechanism whereby the equation can be solved; that is, its root may be determined.

4.5 AXIOMS FOR SOLVING EQUATIONS

The equality sign (=) that separates two equal quantities allows certain operations to be performed in our efforts to solve an equation. For any equation, the following statements are always true:

- Axiom 1. If the same quantity is added to both sides of an equation, the new equation will still be true. If 5 is added to both sides of the equation $x - 5 = 8$, the new equation $x = 13$ results.
- Axiom 2. If the same quantity is subtracted from both sides of an equation, the new equation will still be true. If 5 is subtracted from both sides of the equation $x + 5 = 8$, the new equation $x = 3$ results.

- Axiom 3. If both sides of an equation are multiplied by the same quantity, the new equation will still be true. If both sides of the equation $(1/3)x = 5$ are multiplied by 3, the new equation $x = 15$ results.
- Axiom 4. If both sides of an equation are divided by the same quantity, the new equation will still be true. If both sides of the equation $3x = 9$ are divided by 3, the new equation $x = 3$ results.

The four axioms for solving algebraic equations may be summarized by one general principle. Whatever operation is performed on one side of an equation, the same operation must be performed on the other side of the equation if the equation is to remain true. For example:

1. Solve the equation $4x + 3 = 19$.

Step 1. Using Axiom 2, subtract 3 from both sides of the equation.

$4x + 3 - 3 = 19 - 3$

$4x = 16$

Step 2. Using Axiom 4, divide both sides of the equation by 4.

$(4/4)X - 8 = 2$

2. Solve the equation.

$(1/4)X - 8 = 2$

Step 1. Using Axiom 1, add 8 to both sides of the equation.

$\frac{1}{4}X - 8 + 8 = 2 + 8$

$\frac{1}{4}X = 10$

Step 2. Using Axiom 3, multiply both sides of the equation by 4.

$(4)\frac{1}{4}X = 10(4)$

$X = 40$

In each of these examples, the axioms are used to isolate the unknown on one side of the equation.

There is a shorter method available for applying the addition and subtraction Axioms 1 and 2 to algebraic equations. Any term may be transposed or transferred from one side of an equation to the other provided its sign is changed. For example:

In the equation $5x + 4 = 7$, the 4 can be transposed to the other side of the equation by changing the sign.

$5x + 4 = 7$

$5x = 7 - 4$

$5x = 3$

Notes

This corresponds to applying Axiom 2, subtracting 4 from both sides of the equation.

4.5.1 Solving Fractional Equations

All equations studied so far contained only whole numbers. In addition to whole numbers, an equation may also contain fractions, either common fractions or decimal fractions. A fractional equation is an equation that contains fractions. The unknown may be located anywhere in the equation.

$$\frac{2X+6}{3X} = 5X - \frac{1}{2}$$

is a fractional equation.

Fractional equations are solved exactly as any other equation, using the four axioms. Generally, the equation is first cleared of the fractions. As shown below, operations are performed that remove all of the fractions.

Solve the following fractional equation:

$$\frac{5}{8} - \frac{9}{x} = \frac{1}{2} - \frac{4}{3x}$$

Step 1: Multiply each term by 8x, which is the lowest common denominator.

$$(8x)(\frac{5}{8}) - (8x)(\frac{9}{x}) = (8x)(\frac{1}{2}) - (8x)(\frac{3}{4}x)$$

5x – 72 = 4x - 6

Step 2: Transpose (- 72) and (4x).

5x – 4x = -6 + 72

x = 66

As the final step in the solution of any equation, the root should be substituted back into the equation to ensure that it makes the equation true. If the root does not check, then an error has been made during the solution.

Check to see if x = 66 is a root of the equation.

$$\frac{5}{8} - \frac{9}{x} = \frac{1}{2} - \frac{3}{4x} \qquad \text{Substitute } x = 66$$

$$\frac{5}{8} - \frac{9}{66} = \frac{1}{2} - \frac{3}{(4)(66)} = \frac{(5)(66)-(9)(8)}{(8)(66)} = \frac{(4)(66)-6}{(8)(66)}$$

$$330 - 72 = 264 - 6$$
$$258 = 258$$

The root checks.

4.5.2 Ratio and Proportion

The concept of ratio and proportion are one which is used almost every day. If the quantities listed in a recipe will produce servings for eight, it is instinctively known that to make servings for four, each quantity should be cut in half. What has been done is to form a ratio of the number of servings. A ratio is a comparison of two like quantities by division. It is often indicated by a colon (:), although it is really a fraction.

The ratio of \$25 to \$5 is \$25 ÷ \$5 which equals 5. This is often indicated as:

\$25 : \$5 and is read \$25 is to \$5. It can also be written as a fraction:

$$\$25 : \$5 = \frac{\$25}{\$5} = \frac{5}{1}$$

A proportion is a statement of equality between two equal ratios.

$$\frac{\$5}{\$25} = \frac{2\text{ lb}}{10\text{ lb}}$$

or: \$5 : \$25 = 2 lb : 10 lb

Notice that the ratios are comparisons of different amounts of the like quantities, dollars and pounds, respectively. The proportion simply states that the ratio of \$5 to \$25 is equal to the ratio between 2 pounds and 10 pounds. When the proportion is written out in the colon form, there is a relationship between the various terms. The first and fourth terms in a proportion are called the extremes. The second and third terms are called the means. For example:

In the proportional \$5 : \$25 = 2 lb : 10 lb, \$5 and 10 lb are the extremes, \$25 and 2 lb are the means.

In any proportion, the product of the means equals the product of the extremes.

\$5 : \$25 = 2 lb : 10 lb

The product of the means is [\$25] [2 lb] = 50 dollar-pounds.

The product of the extremes is [\$5] [10 lb] = 50 dollar-pounds.

This can be seen clearly if the proportion is written in fractional form.

Notes

$$\frac{\$5}{\$25} = \frac{2\text{ lb}}{10\text{ lb}}$$

1. Multiply the equation by $25.

$$\$5 = \frac{2\text{ lb}}{10\text{ lb}} \times \$25$$

2. Multiply the equation by 10 lb.

$5 x 10 lb = 2 lb x $25

50 $-lb = 50 $-lb

This principle permits finding any missing term in a proportion. For example:

3. Find the missing term in the proportion 5 : x = 4 : 15.

The product of the extreme is: [5] [15] = 75

The product of the means is: [4] [x] = 4x

4x = 75

$$x = \frac{74}{4} = 18\frac{3}{4}$$

This could also be solved in fractional form.

4. Find the missing term in the proportion 5 : x = 4 : 15.

Rewrite as:

$$\frac{5}{x} = \frac{4}{15}$$

$$5 = \frac{4}{15}x$$

$$(5)(15) = 4x$$

$$x = \frac{(5)(15)}{4} = 18\frac{3}{4}$$

The concepts of ratio and proportion are useful in solving problems such as the example below.

If 5 pounds of apples cost 80 cents, how much will 7 pounds cost?Using x for the cost of 7 pounds of apples, the following proportion can be written.

$$\frac{5}{80} = \frac{7}{x}$$

The product of the extremes is (5)(x) = 5x.

The product of the means is (7)(80) = 560.

Equate these two products and solve the resulting equation.

5x = 560

$$\frac{5x}{5} = \frac{560}{5}$$

x = 112

The unit of x is cents. Thus, 7 pounds of apples cost 112 cents or $1.12.

4.6 BASIC APPROACH TO SOLVING ALGEBRAIC WORD PROBLEMS

The problems encountered in everyday life, however, are rarely stated in equation form. These problems are stated in words, and they must be translated into the appropriate mathematical equations in order to find the solution. This is a two-step process:

- Step 1: Write the equation(s) from the information given.
- Step 2: Solve the equation(s).

Methods for accomplishing Step 2 have already been studied. In this section, the methods for accomplishing Step 1 will be presented.

4.6.1 Steps for Solving Algebraic Word Problems

Before attempting to solve any word problem, the problem must be understood completely. It is frequently beneficial to draw a picture of the physical situation described by the problem. The drawing should be labelled with the known and unknown quantities. At the very least, these quantities should be listed in some logical order.

After ensuring that the problem is understood, it can be solved by following these five fundamental steps:

Step 1. Let some letter, such as x, represent one of the unknowns. There will always be a choice of which unknown can be called x. There is no one right choice. For example, suppose that it is given that an orange costs 3 cents more than an apple. If x equals the cost of the orange, the cost of the apple will be x - 3. On the other hand, if x equals the cost of the apple, then the cost of the orange is x + 3. Either choice is correct.

Step 2. Express the other unknowns, using the information given, in terms of x. In doing this, it is helpful to look for certain words that indicate algebraic operations. The words sum and total signify addition; the words difference and less than signify subtraction; the words times and multiples of signify multiplication; the words divided by and per signify division. The words same as and equal to signify equality.

Step 3. Write an equation. Make the equation say in symbols exactly what the problem says in words. This involves reading the problem carefully to understand exactly what is being asked.

Step 4. Solve the equation using the methods discussed in previous sections.

Step 5. Check the solution by substituting it into the equations. For example:

A family took a trip and travelled a total of 820 miles in three days. They drove twice as many miles the second day as on the first. The third day they drove 60 miles less than they did on the second day. Find the distance travelled each day.

Before solving this word problem, exactly what must be known is required to be found, that is, what are the unknowns? For this example, draw a diagram of the problem and write in the known and unknowns.

Now proceed through the following five steps to solve the problem:

Step 1. Let x = the number of miles driven on the first day. x could = the number of miles driven on the second or third day, but for the moment x = the number of miles driven the first day.

Step 2. It is given that they drove twice as many miles on the second day as the first. Therefore, if x is the number of miles driven on the first day, then 2x is the number driven on the second day. It is also given that on the third day, they drove 60 miles less than on the second day. If they drove 2x miles on the second day, they drove 2x - 60 miles on the third day. All of the unknowns have been expressed in terms of x.

Step 3. Write an equation, which relates the unknowns. Total miles driven = miles driven on the first day + miles driven on second day - miles driven on third day. The total miles driven are 820. Therefore, x + (2x) + (2x - 60) = 820

Step 4. Solve the equation. x + 2x + 2x – 60 = 820

5x = 820 + 60

5x = 880

$$x = \frac{880}{5}$$

x = 176 miles

This is the number of miles driven on the first day.

Answers:

First day: x = 176 miles

Second day: 2x = 352 miles

Third day: 2x – 60 = 292 miles

Step 5. Check the answers. x + 2x + 2x – 60 = 820

176 + 352 + 292 = 820

820 = 820

In solving this problem, let x equal the number of miles driven on the first day. There is nothing magical about this choice. Just to illustrate the point, solve the problem in a slightly different manner. Let y be the number of miles driven on the second day. Then $\frac{y}{2}$ will be the miles driven on the first day, and y - 60 will be the miles driven on the third day. The equation will now be:

$$\frac{Y}{2} + Y + (Y - 60) = 820$$

Solve this equation and see that the answer is the same as before. Try another example:

A man is 2 years older than three times his son's age. Ten years from now he will only be twice as old as his son. What are their ages now?

Step 1. Let x = the son's age now.

Step 2. Then, 3x + 2 = the father's age now.

Step 3. The father's age ten years from now will be two times the son's age at that time.

(3x + 2) + 10 = 2(x + 10)

Step 4. (3x + 2) + 10 = 2(x + 10)

3x + 12 = 2x + 20 x = 8

Answers:

Son's age now: x = 8

Father's age now: 3x + 2 - 3(8) + 2 = 26

4.6.2 Problems Involving Money

Algebraic problems involving money can become confusing because there is a tendency to use quantity and value interchangeably. For example, a quantity of 5 dimes has a value of 50 cents, whereas a quantity of 5 nickels has a value of 25 cents. The quantities are the same, but the values are different. The general relationship used to solve algebraic word problems involving money is:

Total Value = Sum of [(Quantity) times (Value per Quantity)]

Example: The total value of five pennies, two nickels, three dimes, and four quarters is:

[(5) x $0.01)] + [(2) x ($0.05)] + [(3) x ($0.10)] + [(4) x ($0.25)] = Total Value

$0.05 + $0.10 + $0.30 + $1.00 = $1.45

"How many dimes?" is a different question than "how much in dimes?"

A total of 6,000 tickets were sold for a basketball game. Tickets were priced at $2.00 and $2.80 each. A total of $14,264 was collected. How many of each price ticket were sold?

Step 1. Let x = the number of $2.00 tickets sold.

Step 2. Then, 6,000 - x = the number of $2.80 tickets sold.

Step 3. Total Collected = (Number of $2.00 Tickets Sold) x ($2.00) + (Number of $2.80 Tickets Sold) x ($2.80)

$14,264 = (x)($2.00) + (6,000 - x)($2.80)

Step 4. $14,264 = 2x + 16,800 - 2.8x

2.8x - 2x = $16,800 - $14,264

0.8x = 2,536

x = 3,170

Answers:

Number of $2.00 Tickets Sold: x = 3,170

Number of $2.80 Tickets Sold: 6,000 - x = 2,830

Step 5. Check:

(3,170)($2.00) + (2,830)($2.80) = $14,264

$6,340 + $ 7,924 = $14,264

$14,264 = $14,264

Of course, the problem could also have been solved by letting x equal the number (quantity) of $2.80 tickets sold. Note that the quantity of tickets sold is different than the value of the tickets.

4.6.3 Problems in Uniform Motion

There is a large variety of problems that involve travel and travel times. In order to solve these problems, a relationship between distance, speed and time is needed. This relationship is:

Distance = (Speed) x (Time)

D = vt

If a car is moving at a uniform speed of 60 miles per hour for two hours, the distance travelled is 120 miles. d = vt d = (60 mph)(2 hr) d = 120 miles

Algebraic word problems involving uniform motion are solved using this general relationship and following the steps for solving any algebraic word problem. This is shown in the example below.

A man takes a trip of 675 miles, part of the trip by train at 60 mph and the rest of the trip by car at 50 mph. If the entire trip takes 12 hours, how far has he travelled by each mode of transportation?

As always, before attempting to solve the problem, make sure what is being asked is understood.

Step 1. Let x = the number of miles travelled by train.

Step 2. 675 miles total are travelled with x miles travelled by train, and the remainder travelled by car. Thus, the number of miles travelled by car is (675 - x).

The number of hours travelled by train equals the distance travelled divided by the speed. Thus, the number of hours travelled by train equals

$$\frac{x \text{ Miles}}{60 \text{ mph}}$$

The total number of hours travelled is 12 with $\frac{x}{60}$ hours travelled by train and the remainder travelled by car. Thus, the total number of hours travelled by car is:

$$12 - \frac{x}{60}$$

and the distance travelled by car is:

$$50\frac{(12-x)}{60} = 675$$

Step 3. Distance travelled by train + Distance travelled by car = 675.

$$x + (50)\frac{(12-x)}{60} = 675$$

$$x + 600 - \frac{5}{6}x = 675$$

$$\frac{1}{6}x = 75$$

Step 4. Distance travelled by train: x = 450 miles

Distance travelled by car: 675 - x = 675 - 450 = 225 miles.

Step 5. Check

450 + 225 = 675 this corresponds to the total distance travelled.

675 = 675

This example may seem complicated, but if the work is arranged in a logical, step-by-step manner, the solution process becomes straightforward.

4.7 LETS SUM UP

Elementary algebra encompasses some of the basic concepts of algebra, one of the main branches of mathematics. It is typically taught to secondary school students and builds on their understanding of arithmetic. Whereas arithmetic deals with specified numbers, algebra introduces quantities without fixed values, known as variables. This use of variables entails a use of algebraic notation and an understanding of the general rules of the operators introduced in arithmetic. Unlike abstract algebra, elementary algebra is not concerned with algebraic structures outside the realm of real and complex numbers.

The use of variables to denote quantities allows general relationships between quantities to be formally and concisely expressed, and thus enables solving a broader scope of problems. Many quantitative relationships in science and mathematics are expressed as algebraic equations.

4.8 KEYWORD

Uniform motion

Arithmetic

Operation

Operator

Straightforward

Ratio

Proportion

Fractional

Algebraic

Quadratic

4.8 QUESTIONS FOR REVIEW

Q.1 Explain Steps for Solving Algebraic Word Problems.

Q.2 Discuss the use of Parentheses with examples.

Q.3 What are signed numbers?

Q.4 Explain Algebraic expression & terms.

4.9 SUGGESTED READING & REFERENCE

1. William Smyth, *Elementary algebra: for schools and academies*, Publisher Bailey and Noyes, 1864, "Algebraic Operations"
2. ^ Horatio Nelson Robinson, *New elementary algebra: containing the rudiments of science for schools and academies*, Ivison, Phinney, Blakeman, & Co., 1866, page 7
3. ^ Sin Kwai Meng, Chip Wai Lung, Ng Song Beng, "Algebraic notation", in *Mathematics Matters Secondary 1 Express Textbook*, Publisher Panpac Education Pte Ltd, ISBN 9812738827, 9789812738820, page 68
4. ^ William P. Berlinghoff, Fernando Q. Gouvêa, *Math through the Ages: A Gentle History for Teachers and Others*, Publisher MAA, 2004, ISBN 0883857367, 9780883857366, page 75
5. ^ Ramesh Bangia, *Dictionary of Information Technology*, Publisher Laxmi Publications, Ltd., 2010, ISBN 9380298153, 9789380298153, page 212

6. ^ George Grätzer, *First Steps in LaTeX*, Publisher Springer, 1999, ISBN 0817641327, 9780817641320, page 17
7. ^ S. Tucker Taft, Robert A. Duff, Randall L. Brukardt, Erhard Ploedereder, Pascal Leroy, *Ada 2005 Reference Manual*, Volume 4348 of Lecture Notes in Computer Science, Publisher Springer, 2007, ISBN 3540693351, 9783540693352, page 13
8. ^ C. Xavier, *Fortran 77 And Numerical Methods*, Publisher New Age International, 1994, ISBN 812240670X, 9788122406702, page 20
9. ^ Randal Schwartz, brian foy, Tom Phoenix, *Learning Perl*, Publisher O'Reilly Media, Inc., 2011, ISBN 1449313140, 9781449313142, page 24
10. ^ Matthew A. Telles, *Python Power!: The Comprehensive Guide*, Publisher Course Technology PTR, 2008, ISBN 1598631586, 9781598631586, page 46

4.10 ANSWERS TO CHECK YOUR PROGRESS

Check in progress -1

1. Hint: Please refer to section 4.1

2. Hint: Please refer to section 4.2

UNIT - 5 : SEQUENCE AND CONVERGENCE

STRUCTURE

5.1 INTRODUCTION

Let's start off this section with a discussion of just what a sequence is. A sequence is nothing more than a list of numbers written in a specific order. The list may or may not have an infinite number of terms in them although we will be dealing exclusively with infinite sequences in this class. General sequence terms are denoted as follows,

$$\begin{aligned} &a_1 - \text{first term} \\ &a_2 - \text{second term} \\ &\qquad\vdots \\ &a_n - n^{th} \text{ term} \\ &a_{n+1} - (n+1)^{\text{st}} \text{ term} \end{aligned}$$

Because we will be dealing with infinite sequences each term in the sequence will be followed by another term as noted above. In the notation above we need to be very careful with the subscripts. The subscript of $\eta + 1$ denotes the next term in the sequence and NOT one plus the η^{th} term! In other words,

$$a_{n+1} \neq a_n + 1$$

So be very careful when writing subscripts to make sure that the "+1" This is an easy mistake to make when you first start dealing with this kind of thing.

There is a variety of ways of denoting a sequence. Each of the following is equivalent ways of denoting a sequence

$$\{a_1, a_2, \ldots, a_n, a_{n+1}, \ldots\} \qquad \{a_n\} \qquad \{a_n\}_{n=1}^{\infty}$$

.

In the second and third notations above a_n is usually given by a formula.

First, note the difference between the second and third notations above. If the starting point is not important or is implied in some way by the problem it is often not written down as we did in the third notation. Next, we used a starting point of η =1 in the third notation only so we could write one down. There is absolutely no reason to believe that a sequence will start at η =1. A sequence will start where ever it needs to start.

Example 1 **:**Write down the first few terms of each of the following sequences.

(a) $\left\{ \frac{n+1}{n^2} \right\}_{n=1}^{\infty}$

(b) $\left\{ \frac{(-1)^{n+1}}{2^n} \right\}_{n=0}^{\infty}$

(c) $\{b_n\}_{n=1}^{\infty}$, where $b_n = n^{th}$ digit of π

(a) $\left\{ \frac{n+1}{n^2} \right\}_{n=1}^{\infty}$

To get the first few sequence terms here all we need to do is plug in values of n into the formula given and we'll get the sequence terms.

$$\left\{\frac{n+1}{n^2}\right\}_{n=1}^{\infty} = \left\{\underbrace{2}_{n=1}, \underbrace{\frac{3}{4}}_{n=2}, \underbrace{\frac{4}{9}}_{n=3}, \underbrace{\frac{5}{16}}_{n=4}, \underbrace{\frac{6}{25}}_{n=5}, \ldots\right\}$$

Note the inclusion of the "…" at the end! This is an important piece of notation as it is the only thing that tells us that the sequence continues on and doesn't terminate at the last term.

(b) $\left\{\frac{(-1)^{n+1}}{2^n}\right\}_{n=0}^{\infty}$

This one is similar to the first one. The main difference is that this sequence doesn't start at n=1.

$$\left\{\frac{(-1)^{n+1}}{2^n}\right\}_{n=0}^{\infty} = \left\{-1, \frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \ldots\right\}$$

Note that the terms in this sequence alternate in signs. Sequences of this kind are sometimes called alternating sequences.

(c) $\{b_n\}_{n=1}^{\infty}$, where $b_n = n^{th}$ digit of π

This sequence is different from the first two in the sense that it doesn't have a specific formula for each term. However, it does tell us what each term should be. Each term should be the n^{th} digit of p. So we know that π=3.14159265359…

The sequence is then,

{3,1,4,1,5,9,2,6,5,3,5,…}

In the first two parts of the previous example note that we were really treating the formulas as functions that can only have integers plugged into them. Or,

$$f(n) = \frac{n+1}{n^2} \qquad g(n) = \frac{(-1)^{n+1}}{2^n}$$

This is an important idea in the study of sequences (and series). Treating the sequence terms as function evaluations will allow us to do many things with sequences that we couldn't do otherwise. Before delving further into this idea however we need to get a couple more ideas out of the way.

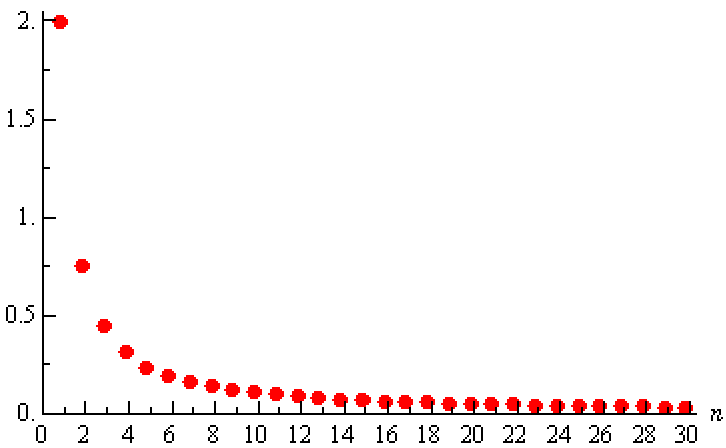
First, we want to think about "graphing" a sequence. To graph the sequence $\{a_n\}$ we plot the points (n, a_n) as n ranges over all possible values on a graph. For instance, let's graph the sequence

$$\left\{\frac{n+1}{n^2}\right\}_{n=1}^{\infty}$$

The first few points on the graph are,

$$(1,2), \left(2,\frac{3}{4}\right), \left(3,\frac{4}{9}\right), \left(4,\frac{5}{16}\right), \left(5,\frac{6}{25}\right), \ldots$$

The graph, for the first 30 terms of the sequence, is then,



This graph leads us to an important idea about sequences. Notice that as n increases the sequence terms in our sequence, in this case, get closer and closer to zero. We then say that zero is the limit (or sometimes the limiting value) of the sequence and write,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{n^2} = 0$$

If you recall, we said earlier that we could think of sequences as functions in some way and so this notation shouldn't be too surprising.

Using the ideas that we developed for limits of functions we can write down the following working definition for limits of sequences.

5.2 DEFINITION OF LIMIT

We say that $\lim_{n\to\infty} a_n = L$ if we can make an as close to L as we want for all sufficiently large n. In other words, the value of the a_n approach L as n approaches infinity.

We say that $\lim_{n\to\infty} a_n = \infty$ if we can make an as large as we want for all sufficiently large n. Again, in other words, the value of the a_n's get larger and larger without bound as n approaches infinity.

We say that $\lim_{n\to\infty} a_n = -\infty$ if we can make a_n as large and negative as we want for all sufficiently large n. Again, in other words, the value of the a_n's are negative and get larger and larger without bound as n approaches infinity.

The working definitions of the various sequence limits are nice in that they help us to visualize what the limit actually is. Just like with limits of functions however, there is also a precise definition for each of these limits. Let's give those before proceeding approaches infinity.

Precise Definition of Limit

1. We say that $\lim_{n\to\infty} a_n = L$ if for every number $\varepsilon > 0$ there is an integer N such that

$$|a_n - L| < \varepsilon \qquad \text{whenever} \qquad n > N$$

2. We say that $\lim_{n\to\infty} a_n = \infty$ if for every number $M > 0$ there is an integer N such that

$$a_n > M \qquad \text{whenever} \qquad n > N$$

3. We say that $\lim_{n\to\infty} a_n = -\infty$ if for every number $M < 0$ there is an integer N such that

$$a_n < M \qquad \text{whenever} \qquad n > N$$

Note that both definitions tell us that in order for a limit to exist and have a finite value all the sequence terms must be getting closer and closer to that finite value as n increases. Now that we have the definitions of the limit of sequences out of the way we have a bit of terminology that we need to look at. If $\lim_{n\to\infty} a_n$ exists and is finite we say that the sequence is **convergent**. If $\lim_{n\to\infty} a_n$ doesn't exist or is infinite we say the sequence **diverges**. Note that sometimes we will say the sequence **diverges to** ∞ if $\lim_{n\to\infty} a_n = \infty$ and if $\lim_{n\to\infty} a_n = -\infty$ we will sometimes say that the sequence diverges to $-\infty$.

Most limits of most sequences can be found using one of the following theorems.

Theorem 1

Given the sequence $\{a_n\}$ if we have a function $f(x)$ such that $f(n) = a_n$ and $\lim_{x\to\infty} f(x) = L$ then $\lim_{n\to\infty} a_n = L$

This theorem is basically telling us that we take the limits of sequences much like we take the limit of functions. In fact, in most cases we'll not even really use this theorem by explicitly writing down a function. We will more often just treat the limit as if it were a limit of a function and take the limit as we always did back in Calculus I when we were taking the limits of functions.

So, now that we know that taking the limit of a sequence is nearly identical to taking the limit of a function we also know that all the properties from the limits of functions will also hold.

Properties

If $\{a_n\}$ and $\{b_n\}$ are both convergent sequences then,

1. $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$
2. $\lim_{n\to\infty} c a_n = c \lim_{n\to\infty} a_n$
3. $\lim_{n\to\infty}(a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$
4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, provided $\lim_{n\to\infty} b_n \neq 0$
5. $\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p$ provided $a_n \geq 0$

T

hese properties can be proved using Theorem 1 above and the function limit properties we saw in Calculus I or we can prove them directly using the precise definition of a limit using nearly identical proofs of the function limit properties.

Next, just as we had a Squeeze Theorem for function limits we also have one for sequences and it is pretty much identical to the function limit version.

Squeeze Theorem for Sequences

If $a_n \le c_n \le b_n$ for all $n > N$ for some N and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = L$ then $\lim\limits_{n\to\infty} c_n = L$.

Note that in this theorem the "for all n>**N** for some **N**" is really just telling us that we need to have $a_n \le c_n \le b_n$ for all sufficiently large n, but if it isn't true for the first few n that won't invalidate the theorem. As we'll see not all sequences can be written as functions that we can actually take the limit of. This will be especially true for sequences that alternate in signs. While we can always write these sequence terms as a function we simply don't know how to take the limit of a function like that. The following theorem will help with some of these sequences.

Theorem 2

If $\lim\limits_{n\to\infty} |a_n| = 0$ then $\lim\limits_{n\to\infty} a_n = 0$.

Note that in order for this theorem to hold the limit MUST be zero and it won't work for a sequence whose limit is not zero. This theorem is easy enough to prove so let's do that.

Proof of Theorem 2

The main thing to this proof is to note that,

$$-|a_n| \le a_n \le |a_n|$$

Then note that,

$$\lim_{n\to\infty} (-|a_n|) = -\lim_{n\to\infty} |a_n| = 0$$

We then have $\lim_{n\to\infty}(-|a_n|) = \lim_{n\to\infty}|a_n| = 0$ and so by the Squeeze Theorem we must also have,

$$\lim_{n\to\infty} a_n = 0$$

5.3 SQUEEZE THEOREM

The sequence $\{r^n\}_{n=0}^{\infty}$ converges if $-1 < r \le 1$ and diverges for all other values of r. Also,

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

The next theorem is a useful theorem giving the convergence/divergence and value (for when it's convergent) of a sequence that arises on **occasion.**

Theorem 3

Here is a quick (well not so quick, but definitely simple) partial proof of this theorem.

Partial Proof of Theorem 3

We'll do this by a series of cases although the last case will not be completely proven.

Case 1 : $r>1$

We know from Calculus I that $\lim_{x\to\infty} r^x = \infty$ if $r > 1$ and so by Theorem 1 above we also know that $\lim_{n\to\infty} r^n = \infty$ and so the sequence diverges if r>1.

Case 2 : r=1

In this case we have, $\lim_{n\to\infty} r^n = \lim_{n\to\infty} 1^n = \lim_{n\to\infty} 1 = 1$

So, the sequence converges for r=1 and in this case its limit is 1.

Case 3 : 0<r<1

We know from Calculus I that $\lim_{x\to\infty} r^x = 0$ if $0 < r < 1$

and so by Theorem 1 above we also know that
$\lim_{n\to\infty} r^n = 0$ and so the sequence converges if $0 < r < 1$

and in this case its limit is zero.

Case 4 : r=0

In this case we have, $\lim_{n\to\infty} r^n = \lim_{n\to\infty} 0^n = \lim_{n\to\infty} 0 = 0$

So, the sequence converges for r=0 and in this case its limit is zero.

Case 5 : −1<r<0
First let's note that if −1<r<0 then 0<|r|<1

then by Case 3 above we have,

$$\lim_{n\to\infty} |r^n| = \lim_{n\to\infty} |r|^n = 0$$

Theorem 2 above now tells us that we must also have,

$\lim_{n\to\infty} r^n = 0$ and so if $-1 < r < 0$ the sequence converges and has a limit of 0

Case 6 : r =−1

In this case the sequence is,

$$\{r^n\}_{n=0}^{\infty} = \{(-1)^n\}_{n=0}^{\infty} = \{1, -1, 1, -1, 1, -1, 1, -1, \ldots\}_{n=0}^{\infty}$$

and hopefully it is clear that $\lim_{n\to\infty} (-1)^n$ doesn't exist. Recall that in order of this limit to exist the terms must be approaching a single value as n increases. In this case however the terms just alternate between 1 and -1 and so the limit does not exist. So, the sequence diverges for r = −1

Case 7 : r <−1

In this case we're not going to go through a complete proof. Let's just see what happens if we let r = −2 for instance. If we do that the sequence becomes,

$$\{r^n\}_{n=0}^{\infty} = \{(-2)^n\}_{n=0}^{\infty} = \{1, -2, 4, -8, 16, -32, \ldots\}_{n=0}^{\infty}$$

So, if r = −2 we get a sequence of terms whose values alternate in sign and get larger and larger and so $\lim_{n\to\infty} (-2)^n$ doesn't exist. It does not settle

Notes

down to a single value as n increases nor do the terms ALL approach infinity. So, the sequence diverges for r = −2.

We could do something similar for any value of r such that r<−1 and so the sequence diverges for r<−1.

Let's take a look at a couple of examples of limits of sequences.

Example 2 Determine if the following sequences converge or diverge. If the sequence converges determine its limit.

(a) $\left\{\frac{3n^2-1}{10n+5n^2}\right\}_{n=2}^{\infty}$

(b) $\left\{\frac{e^{2n}}{n}\right\}_{n=1}^{\infty}$

(c) $\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$

(d) $\left\{(-1)^n\right\}_{n=0}^{\infty}$

(a) $\left\{\frac{3n^2-1}{10n+5n^2}\right\}_{n=2}^{\infty}$

In this case all we need to do is recall the method that was developed in Calculus I to deal with the limits of rational functions. To do a limit in this form all we need to do is factor from the numerator and denominator the largest power of n, cancel and then take the limit.

$$\lim_{n\to\infty}\frac{3n^2-1}{10n+5n^2}=\lim_{n\to\infty}\frac{n^2\left(3-\frac{1}{n^2}\right)}{n^2\left(\frac{10}{n}+5\right)}=\lim_{n\to\infty}\frac{3-\frac{1}{n^2}}{\frac{10}{n}+5}=\frac{3}{5}$$

So, the sequence converges and its limit is $\frac{3}{5}$

(b) $\left\{\frac{e^{2n}}{n}\right\}_{n=1}^{\infty}$

We will need to be careful with this one. We will need to use L'Hospital's Rule on this sequence. The problem is that L'Hospital's Rule only works on functions and not on sequences. Normally this would be a problem, but we've got Theorem 1 from above to help us out. Let's define

$$f(x)=\frac{e^{2x}}{x}$$

and note that,

$$f(n) = \frac{\mathbf{e}^{2n}}{n}$$

Theorem 1 says that all we need to do is take the limit of the function.

$$\lim_{n\to\infty} \frac{\mathbf{e}^{2n}}{n} = \lim_{x\to\infty} \frac{\mathbf{e}^{2x}}{x} = \lim_{x\to\infty} \frac{2\mathbf{e}^{2x}}{1} = \infty$$

So, the sequence in this part diverges (to ∞).

More often than not we just do L'Hospital's Rule on the sequence terms without first converting to x's since the work will be identical regardless of whether we use x or n. However, we really should remember that technically we can't do the derivatives while dealing with sequence terms.

(c) $\left\{ \frac{(-1)^n}{n} \right\}_{n=1}^{\infty}$

We will also need to be careful with this sequence. We might be tempted to just say that the limit of the sequence terms is zero (and we'd be correct). However, technically we can't take the limit of sequences whose terms alternate in sign, because we don't know how to do limits of functions that exhibit that same behavior. Also, we want to be very careful to not rely too much on intuition with these problems. As we will see in the next section, and in later sections, our intuition can lead us astray in these problems if we aren't careful.

So, let's work this one by the book. We will need to use Theorem 2 on this problem. To this we'll first need to compute,

$$\lim_{n\to\infty} \left| \frac{(-1)^n}{n} \right| = \lim_{n\to\infty} \frac{1}{n} = 0$$

Therefore, since the limit of the sequence terms with absolute value bars on them goes to zero we know by Theorem 2 that,

$$\lim_{n\to\infty}\frac{(-1)^n}{n}=0$$

which also means that the sequence converges to a value of zero.

(d) $\left\{(-1)^n\right\}_{n=0}^{\infty}$

For this theorem note that all we need to do is realize that this is the sequence in Theorem 3 above using r = −1. So, by Theorem 3 this sequence diverges.

We now need to give a warning about misusing Theorem 2. Theorem 2 only works if the limit is zero. If the limit of the absolute value of the sequence terms is not zero then the theorem will not hold. The last part of the previous example is a good example of this (and in fact this warning is the whole reason that part is there). Notice that

$$\lim_{n\to\infty}\left|(-1)^n\right|=\lim_{n\to\infty}1=1$$

and yet, $\lim_{n\to\infty}(-1)^n$ doesn't even exist let alone equal 1. So, be careful using this Theorem 2. You must always remember that it only works if the limit is zero.

Theorem 4

For the sequence {an} if both ,

$\lim_{n\to\infty}a_{2n}=L$ and $\lim_{n\to\infty}a_{2n+1}=L$ then $\{a_n\}$ is convergent and $\lim_{n\to\infty}a_n=L$.

Proof of Theorem 4

Let ε>0. Then since

$\lim_{n\to\infty}a_{2n}=L$ there is an $N_1>0$ such that if $n>N_1$ we know that,

$$|a_{2n}-L|<\varepsilon$$

Likewise, because

$\lim_{n\to\infty} a_{2n+1} = L$ there is an $N_2 > 0$ such that if $n > N_2$ we know that,
$|a_{2n+1} - L| < \varepsilon$

Now, let

$N = \max\{2N_1, 2N_2 + 1\}$ and let $n > N$. Then either $a_n = a_{2k}$ for some $k > N_1$ or $a_n = a_{2k+1}$ for some $k > N_2$

and so in either case we have that,

$$|a_n - L| < \varepsilon$$

Therefore,

$\lim_{n\to\infty} a_n = L$ and so $\{a_n\}$ is convergent.

CHECK YOUR PROGRESS-1

Q. 1 State and prove Squeez's Theorem

………………………………………………………………………………

………………………………………………………………………………

………………………………………………………………………………

………………………………………………………………………………

Q.2 Define limits and state its properties.

………………………………………………………………………………

………………………………………………………………………………

………………………………………………………………………………

………………………………………………………………………………

5.4 SEQUENCE & CONVERGENCE

Sequences are, basically, countable many numbers arranged in an order that may or may not exhibit certain patterns. Here is the formal definition of a sequence:

Definition: Sequence

A sequence of real numbers is a function f: N → R. In other words, a sequence can be written as f(1), f(2), f(3), ... Usually, we will denote such a sequence by the symbol $\{a_j\}_{j=1}^{\infty}$, where aj = f(j).

For example, the sequence 1, 1/2, 1/3, 1/4, 1/5 ... is written as $\{1/j\}_{j=1}^{\infty}$. Keep in mind that despite the strange notation, a sequence can be thought of as an ordinary function. In many cases that may not be the most expedient way to look at the situation. It is often easier to simply look at a sequence as a 'list' of numbers that may or may not exhibit a certain pattern.

We now want to describe what the long-term behaviour, or pattern, of a sequence is, if any.

Definition: Convergence

A sequence $\{a_j\}_{j=1}^{\infty}$ of real (or complex) numbers is said to converge to a real (or complex) number c if for every ϵ > 0 there is an integer N > 0 such that if j > N then

| aj - c | < ϵ

The number c is called the limit of the sequence $\{a_j\}_{j=1}^{\infty}$ and we sometimes write aj → c.

If a sequence $\{a_j\}_{j=1}^{\infty}$ does not converge, then we say that it **diverges**.

The sequence $\{1/j\}_{j=1}^{\infty}$ converges to zero.

$$\{1/j\}_{j=1}^{\infty} = \{1,\ 1/2,\ 1/3,\ 1/4,\ \dots\},$$

which seems to indicate that the terms are getting closer and closer to zero. According to the definition of convergence, we need to show that no matter which $\epsilon > 0$ one chooses, the sequence will eventually become smaller than this number. To be precise: take any $\epsilon > 0$. Then there exists a positive integer N such that $1 / N < \epsilon$. Therefore, for any $j > N$ we have:

$$| 1/j - 0 | = | 1/j | < 1/N < \varepsilon$$

whenever $j>N$. But this is precisely the definition of the sequence $\{1/j\}$ converging to zero.

While it looks like this proof is easy, it is a good indication for 'ε-arguments' that will appear again and again. In most of those cases the proper choice of N will make it appear as if the proof works like magic.

The sequence $\{(-1)^j\}_{j=1}^{\infty}$ does not converge.

Note that $\{(-1)^j\}_{j=1}^{\infty} = \{-1, 1, -1, 1, -1, 1, \ldots\}$.

While this sequence does exhibit a definite pattern, it does not get close to any one number, i.e. it does not seem to have a limit. Of course we must prove this statement, so we will use a proof by contradiction.

Suppose that the sequence did converge to a limit L. Then, for $\varepsilon = 1/2$ there exists a positive integer N such that

$$| (-1)^n - L | < 1/2$$

for all $n > N$. But then, for some $n > N$, we have the inequality:

$$2 = | (-1)^{n+1} - (-1)^n | = | ((-1)^{n+1} - L) + (L - (-1)^n) |$$
$$\leq | (-1)^{n+1} - L | + | (-1)^n - L | < 1/2 + 1/2 = 1$$

for $n > N$, which is a contradiction since it says that $2 < 1$, which is not true.

The sequence $\left\{\frac{n}{2^n}\right\}_{n=0}^{\infty}$ converges to zero.

$\{ n / 2^n \} = \{ 0, 1/2, 1/2, 3/8, 1/4, 5/32, \ldots\}$. It is not clear, but it seems as if the terms get smaller and smaller. Indeed this is the case, and we will prove it:

First, we can use induction to show that

$$n^2 \leq 2^n$$

for $n > 3$. But then we have that

$$n^2 / 2^n \leq 1$$

or equivalently

Notes

$n / 2^n \leq 1/n$

for $n > 3$. But now you should be able to finish the proof yourself. As a hint, for a given ε, choose

$N = max\{3, 1/\varepsilon\}$

Convergent sequences, in other words, exhibit the behavior that they get closer and closer to a particular number. Note, however, that divergent sequence can also have a regular pattern, as in the second example above. But it is convergent sequences that will be particularly useful to us right now.

We are going to establish several properties of convergent sequences, most of which are probably familiar to you. Many proofs will use an ' ε argument' as in the proof of the next result. This type of argument is not easy to get used to, but it will appear again and again, so that you should try to get as familiar with it as you can.

Convergent Sequences are bounded

Let $\{a_j\}_{j=1}^{\infty}$ be a convergent sequence. Then the sequence is bounded, and the limit is unique.

Proof:

Let's prove uniqueness first. Suppose the sequence has two limits, a and a'. Take any $\varepsilon > 0$. Then there is an integer N such that:

$| a_j - a | < \varepsilon$

if $j > N$. Also, there is another integer N' such that

$| a_j - a' | < \varepsilon$

if $j > N'$. Then, by the triangle inequality:

$$\begin{aligned} | a - a' | &= | a - a_j + a_j - a' | \\ &\leq |a_j - a | + | a_j - a' | \\ &< \varepsilon + \varepsilon = 2\varepsilon \end{aligned}$$

if $j > max\{N,N'\}$. Hence $| a - a' | < 2\varepsilon$ for any $\varepsilon > 0$. But that implies that $a = a'$, so that the limit is indeed unique.

Next, we prove boundedness. Since the sequence converges, we can take, for example, $\varepsilon = 1$. Then

| aj - a | < 1

if j > N. Fix that number N. We have that

| aj | ≤ | aj - a | + | a | < 1 + |a|

for all j > N. Define

M = max{|a1|, |a2|, ..., |aN|, (1 + |a|)}

Then | aj | < M for all j, i.e. the sequence is bounded as required.

Example :

The Fibonacci numbers are recursively defined as $x_1 = 1$, $x_2 = 1$, and for all $n > 2$ we set $x_n = x_{n-2} + x_{n-1}$. The sequence of Fibonacci numbers *{1, 1, 2, 3, 5 ...}* does not converge.

We will show by induction that the sequence of Fibonacci numbers is unbounded. If that is true, then the sequence cannot converge, because every convergent sequence must be bounded.

As for the induction process: The first terms of the Fibonacci numbers are

{1, 1, 2, 3, 5, 8, 13, 21 ...}

We will show that the *n*th term of that sequence is greater or equal to *n*, at least for $n > 4$.

Property Q(n):

$x_n \geq n$ for all $n > 4$

Check Q(5) (the lowest term):

$x_5 = x_4 + x_3 = 3 + 2 = 5 \geq 5$ is true.

Assume Q(n) true:

$x_n \geq n$ for all $n > 4$

Check Q(n+1):

$x_{n+1} = x_n + x_{n-1} \geq n + x_{n-1} \geq n + 1 \geq n$

Hence, by induction the Fibonacci numbers are unbounded and the sequence cannot converge.

Convergent sequences can be manipulated on a term by term basis, just as one would expect:

Algebra on Convergent Sequences

Suppose $\{a_j\}_{j=1}^{\infty}$ and $\{b_j\}_{j=1}^{\infty}$ are converging to a and b, respectively. Then

1. **Their sum is convergent to $a + b$, and the sequences can be added term by term.**
2. **Their product is convergent to $a * b$, and the sequences can be multiplied term by term.**
3. **Their quotient is convergent to a / b, provide that $b \# 0$, and the sequences can be divided term by term (if the denominators are not zero).**
4. **If $a_n \leq b_n$ for all n, then $a \leq b$**

The proofs of these statements involve the triangle inequality, as well as an occasional trick of adding and subtracting zero, in a suitable form. A proof of the first statement, for example, goes as follows.

Take any $\varepsilon > 0$. We know that $a_n \to a$, which implies that there exists an integer N_1 such that

$$| a_n - a | < \varepsilon / 2$$

if $n > N_1$. Similarly, since $b_n \to b$ there exists another integer N_2 such that

$$| b_n - b | < \varepsilon / 2$$

if $n > N_2$. But then we know that

$$| (a_n + b_n) - (a + b) | = | (a_n - a) + (b_n - b) |$$
$$\leq | a_n - a | + | b_n - b |$$
$$< \varepsilon /2 + \varepsilon /2 = \varepsilon$$

if $n > max(N_1, N_2)$, which proves the first statement.

Proving the second statement is similar, with some added tricks. We know that $\{ b_n \}$ converges, therefore there exists an integer N_1 such that

$$| b_n | < |b| + 1$$

if $n > N_1$. We also know that we can find integers N_2 and N_3 so that

$$| a_n - a | < \varepsilon / (|b| + 1)$$

if $n > N_2$, and

$| b_n - b | < \varepsilon / (|a| + 1)$

if $n > N_3$, because $|a|$ and $|b|$ are some *fixed* numbers. But then we have:

$| a_n b_n - a b | = | a_n b_n - a b_n + a b_n - a b |$

$= | b_n(a_n - a) + a (b_n - b) |$

$\leq | b_n| |a_n - a | + | a | | b_n - b |$

$< (| b | + 1) \varepsilon / (|b| + 1) + | a | \varepsilon / (|a| +1) < 2 \varepsilon$

If $n > max(N_1, N_2, N_3)$, which proves the second statement.

The proof of the third statement is similar, so we will leave it as an exercise.

The last statement does require a new trick: we will use a proof by contradiction to get that result:

Assume that $a_n \leq b_n$ for all n, but $a > b$.

We now need to work out the contradiction: the idea is that since $a > b$ there is some number c such that $b < c < a$.

<----------[b]-------[a]-------->

<----------[b]--[c]--[a]-------->

Since a_n converges to a, we can make the terms of the sequence fall between c and a, and the terms of b_n between b and c. But then we no longer have that $a_n \leq b_n$, which is our contradiction. Now let's formalize this idea:

Let $c = (a + b)/2$. Then clearly $b < c < a$ (verify!). Choose N_1 such that $b_n < c$ if $n > N_1$. That works because $b < c$. Also choose N_2 such that $a_n > c$ if $n > N_2$. But now we have that

$b_n < c < a_n$

for $n > max(N_1, N_2)$. That is a contradiction to the original assumption that $a_n \leq b_n$ for all n. Hence it cannot be true that $a > b$, so that the statement is indeed proved.

This theorem states exactly what you would expect to be true. The proof of it employs the standard trick of 'adding zero' and using the triangle inequality. Try to prove it on your own before looking it up.

Note that the fourth statement is no longer true for strict inequalities. In other words, there are convergent sequences with $a_n < b_n$ for all n, but strict inequality is no longer true for their limits.

While we now know how to deal with convergent sequences, we still need an easy criterion that will tell us whether a sequence converges. The next proposition gives reasonable easy conditions, but will not tell us the actual limit of the convergent sequence.

First, recall the following definitions:

Monotonicity:

A sequence $\{a_j\}_{j=1}^{\infty}$ *is called* ***monotone increasing*** *if* $a_{j+1} \geq a_j$ *for all* j.

A sequence $\{a_j\}_{j=1}^{\infty}$ *is called* ***monotone decreasing*** *if* $a_j \geq a_{j+1}$ *for all* j.

In other words, if every next member of a sequence is larger than the previous one, the sequence is growing or monotone increasing. If the next element is smaller than each previous one, the sequence is decreasing. While this condition is easy to understand, there are equivalent conditions that are often easier to check:

- **Monotone increasing:**
    1. $a_{j+1} \geq a_j$
    2. $a_{j+1} - a_j \geq 0$
    3. $a_{j+1} / a_j \geq 1$, if $a_j > 0$
- **Monotone decreasing:**
    1. $a_{j+1} \leq a_j$
    2. $a_{j+1} - a_j \leq 0$
    3. $a_{j+1} / a_j \leq 1$, if $a_j > 0$

Examples:

Is the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ monotone increasing or decreasing?

One can start to investigate this statement without having to suspect the correct answer. We will simply compare the quotient of two consecutive terms to check whether the answer is greater or less than one:

(1/n) / (1/ (n+1)) = (n+1) / n > 1

Hence, the *n*-th term of the sequence divided by the *(n+1)* term is always greater than 1, or, in other words, the *n*-th term is greater than the *(n+1)*-th term.

That is the definition of a decreasing sequence so that the sequence is decreasing. Checking a graphical representation of this sequence confirms that.

Is the sequence $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$ monotone increasing or decreasing ?

We will again not guess what the correct answer might be ahead of time. We will instead look at the difference between two consecutive terms and see if that comes out greater or less than zero.

$$\frac{n}{n+1}-\frac{n+1}{n+2}=\frac{n(n+2)-(n+1)^2}{(n+1)(n+2)}=\frac{-1}{(n+1)(n+2)}<0$$

This means that the *n*-th term minus the *(n+1)*-th term of the sequence is less than 0, so that the *n*-th term is less than then *(n+1)*-th term.

That means, by definition, that the sequence is increasing. Checking a graphical representation of this sequence confirms that.

Is it true that a bounded sequence converges ? How about monotone increasing sequences ?

Both statements are false. As a counter-example to the first statement, consider the sequence

$\{ (-1)^j \}$

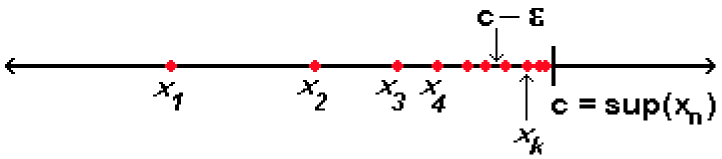
Each term of this sequence is bounded by -1 or +1, so that the sequence is indeed bounded. But, as we have seen before, the sequence does not converge.

As for the second statement, consider the simple sequence *{n}*, i.e. the sequence consisting of the numbers {1, 2, 3, 4, ...}. It is obviously increasing, but does not converge to a finite number.

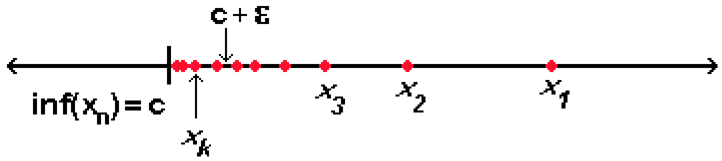
It does, however, get closer and closer to infinity, but we do not, at this time, consider this convergent

Here is a very useful theorem to establish convergence of a given sequence (without, however, revealing the limit of the sequence): First, we have to apply our concepts of supremum and infimum to sequences:

- If a sequence $\{x_k\}_{k=1}^{\infty}$ is bounded above, then $c = sup(x_k)$ is finite. Moreover, given any $\varepsilon > 0$, there exists at least one integer k such that $x_k > c - \varepsilon$, as illustrated in the picture.

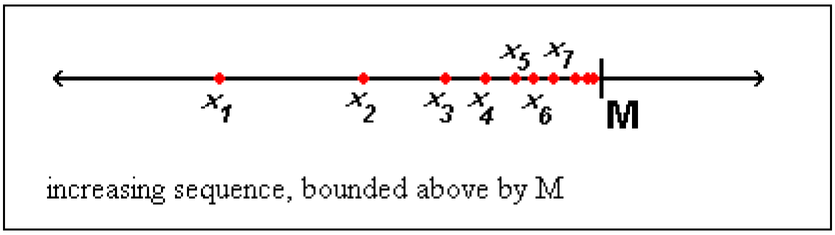


- If a sequence $\{x_k\}_{k=1}^{\infty}$ is bounded below, then c = inf(xk) is finite. Moreover, given any ε > 0, there exists at least one integer k such that xk < c + ε, as illustrated in the picture.

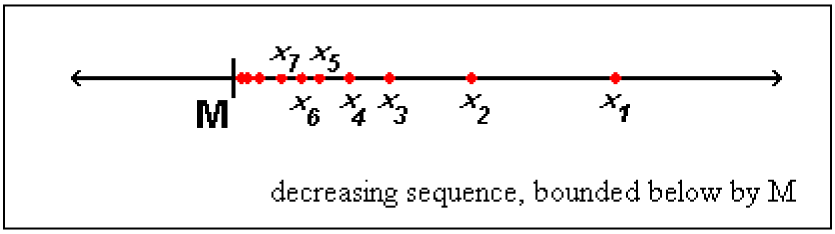


Proposition: Monotone Sequences

If $\{x_k\}_{k=1}^{\infty}$ is a monotone increasing sequence that is bounded above, then the sequence must converge.



If $\{x_k\}_{k=1}^{\infty}$ is a monotone decreasing sequence that is bounded below, then the sequence must converge.



Let's look at the first statement, i.e. the sequence in monotone increasing. Take an $\varepsilon > 0$ and let $c = sup(x_k)$. Then c is finite, and given $\varepsilon > 0$, there exists at least one integer N such that $x_N > c - \varepsilon$. Since the sequence is monotone increasing, we then have that

$x_k > c - \varepsilon$

for all $k > N$, or

$| c - x_k | < \varepsilon$

for all $k > N$. But that means, by definition, that the sequence converges to c.

The proof for the infimum is very similar, and is left as an exercise.

Using this result it is often easy to prove convergence of a sequence just by showing that it is bounded and monotone. The downside is that this method will not reveal the actual limit, just prove that there *is* one.

Example:

The sequences $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ and $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$ both converge.

First, let us consider the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$. It is decreasing because:

$(1/n) - (1 / (n+1)) > 0$

Also, the sequence is bounded below by 0, because each term is positive. Hence, the sequence must converge.

Note that this does not tell us the actual limit. But we have proved before that this sequence converges to 0.

Next, we consider the sequence $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$. This sequence is increasing because

$n / (n+1) - (n+1) / (n + 2) < 0$

The sequence is also bounded above by 1, because $n < n + 1$ so that $n / (n + 1) < 1$

Hence, the sequence must converge.

Note that this does not tell us what the limit of the sequence is. However, the limit is equal to 1, as you can easily prove yourself.

Define $x_1 = b$ and let $x_n = x_{n-1} / 2$ for all $n > 1$. Then this sequence converges for any number b.

The proof is very easy using the theorem on monotone, bounded sequences:

- $b > 0$: the sequence is decreasing and bounded below by 0.
- $b < 0$: the sequence is increasing and bounded above by 0
- $b = 0$: the sequence is constantly equal to zero

In either case the sequence converges. As to finding the actual limit, we proceed as follows: we already know that the limit exists. Call that limit L. Then we have:

$\lim x_n = L = \lim x_{n+1}$

But then we have that

$L = \lim x_{n+1} = \lim x_n / 2 = 1/2 \lim x_n = 1/2\, L$

so that we have the equation for the unknown limit L:

$L = 1/2\, L$

Therefore, the limit must be zero.

This proof illustrates the advantage of knowing that a sequence converges. Based on that fact it was easy to determine the actual limit of this recursively defined sequence. On the other hand, it would be very difficult to try to establish convergence based on the original definition of a convergent sequence

Examples: Computing Square Roots

Let $a > 0$ and $x_0 > 0$ and define the recursive sequence

$x_{n+1} = {}^1/_2 (x_n + {}^a/x_n)$

Show that this sequence converges to the square root of a regardless of the starting point $x_0 > 0$.

Before giving the proof, let's see how this recursive sequence can be used to compute a square root very efficiently. Let's say we want to compute $\sqrt{2}$. Let's start with $x_0 = 2$. Note that the 'true' value of $\sqrt{2}$ with 12 digits after the period is *1.414213562373*. Our recursive sequence would approximate this value quickly as follows:

Term	**Exact Value**	**Approximate Value**
x_0	2	
$x_1 = {}^1/_2 (x_0 + {}^a/x_0)$	${}^1/_2 (2 + {}^2/_2) = {}^3/_2$	1.5
$x_2 = {}^1/_2 (x_1 + {}^a/x_1)$	${}^1/_2 ({}^3/_2 + {}^2/_{3/2}) = {}^{17}/_{12}$	1.416666667
$x_3 = {}^1/_2 (x_2 + {}^a/x_2)$	${}^1/_2 ({}^{17}/_{12} + {}^2/_{17/12}) = {}^{577}/_{408}$	1.414215686
$x_4 = {}^1/_2 (x_3 + {}^a/x_3)$	${}^1/_2 ({}^{577}/_{408} + {}^2/_{577/408}) = {}^{665857}/_{470832}$	1.414213562375

After only 4 steps our sequence has approximated the 'true' value of $\sqrt{2}$ with 11 digits accuracy.

Now that we have seen the usefulness of this particular recursive sequence we need to prove that it really does converge to the square root of *a*.

First, note that $x_n > 0$ for all n so that the sequence is bounded below.

Notes

Next, let's see if the sequence is monotone decreasing, in which case it would have to converge to some limit. Compute

$$x_n - x_{n+1} = x_n - \tfrac{1}{2} (x_n + a / x_n) = \tfrac{1}{2} (x_n^2 - a) / x_n$$

Now let's take a look at $x_n^2 - a$:

$$\begin{aligned} x_n^2 - a &= \tfrac{1}{4} (x_{n-1} + a / x_{n-1})^2 - a \\ &= \tfrac{1}{4} x_{n-1}^2 + \tfrac{1}{2} a + \tfrac{1}{4} a^2 / x_{n-1}^2 - a \\ &= \tfrac{1}{4} x_{n-1}^2 - \tfrac{1}{2} a + \tfrac{1}{4} a^2 / x_{n-1}^2 \\ &= \tfrac{1}{4} (x_{n-1} - a / x_{n-1})^2 \\ &\geq 0 \end{aligned}$$

But that means that $x_n - x_{n+1} \geq 0$, or equivalently $x_n \geq x_{n+1}$. Hence, the sequence is monotone decreasing and bounded below by 0 so it must converge.

We now know that $\lim_{n \to \infty} x_n = L$. To find that limit, we could try the following:

$$(*)\ L = \lim_{n \to \infty} x_n = \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} \tfrac{1}{2} (x_n + a/x_n) = \tfrac{1}{2} (L + a / L)$$

Solving the equation $L = \tfrac{1}{2} (L + a / L)$ gives

$$2 L^2 = L^2 + a$$

or equivalently

$$L^2 = a$$

which means that the limit L is indeed the square root of a, as required.

However, our proof contains one small caveat. In order to take the limit inside the fraction in equation (*) we need to know that L is not zero *before* we can write down equation (*). We already know that x_n is bounded below by zero, but that is not good enough to exclude the possibility of $L = 0$. But we have already shown that

$$x_n^2 - a = \tfrac{1}{4} (x_{n-1} - a / x_{n-1})^2 \geq 0$$

so that $x_n^2 \geq a$. That implies that the limit of the sequence (which we already know exists) is strictly positive since $a > 0$. Therefore equation (*) is justified and we have completed the proof.

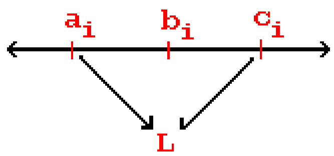
There is one more simple but useful theorem that can be used to find a limit if comparable limits are known. The theorem states that if a sequence is pinched in between two convergent sequences that converge to the same limit, then the sequence in between must also converge to the same limit.

Theorem: The Pinching Theorem

Suppose $\{a_j\}$ and $\{c_j\}$ are two convergent sequences such that $\lim a_j = \lim c_j = L$. If a sequence $\{b_j\}$ has the property that
$a_j \leq b_j \leq c_j$
for all j, then the sequence $\{b_j\}$ converges and $\lim b_j = L$.

Proof:

The statement of the theorem is easiest to memorize by looking at a diagram:



All b_j are between a_j and c_j, and since a_j and c_j converge to the same limit L the b_j have no choice but to also converge to L.

Of course this is not a formal proof, so here we go: we want to show that given any $\varepsilon > 0$ there exists an integer N such that $| b_j - L | < \varepsilon$ if $j > N$. We know that

$a_j \leq b_j \leq c_j$

Subtracting L from these inequalities gives:

$a_j - L \leq b_j - L \leq c_j - L$

But there exists an integer N_1 such that $| a_j - L | < \varepsilon$ or equivalently

$-\varepsilon < a_j - L < \varepsilon$

and another integer N_2 such that $| c_j - L | < \varepsilon$ or equivalently

$-\varepsilon < c_j - L < \varepsilon$

if $j > max(N_1, N_2)$. Taking these inequalities together we get:

$-\varepsilon < a_j - L \leq b_j - L \leq c_j - L < \varepsilon$

But that means that

$-\varepsilon < b_j - L < \varepsilon$

Notes

or equivalently $| b_j - L | < \varepsilon$ as long as $j > max(N_1, N_2)$. But that means that $\{b_j\}$ converges to L, as required.

Examples

Show that the sequence *sin(n) / n* and *cos(n) / n* both converge to zero.

This might seem difficult because trig functions such as *sin* and *cos* are often tricky. However, using the Pinching theorem the proof will be very easy.

We know that $| sin(x) | \leq 1$ for all x. Therefore

$-1 \leq sin(n) \leq 1$

for all n. But then we also know that:

$-1/n \leq sin(n)/n \leq 1/n$

The sequences $\{1/n\}$ and $-1/n$ both converge to zero so that the Pinching theorem applies and the term in the middle must also converge to zero.

To prove the statement involving the *cos* is similar and left as an exercise.

5.5 CAUCHY SEQUENCES

What is slightly annoying for the mathematician (in theory and in praxis) is that we refer to the limit of a sequence in the definition of a convergent sequence when that limit may not be known at all. In fact, more often than not it is quite hard to determine the actual limit of a sequence.

We would prefer to have a definition which only includes the known elements of the particular sequence in question and does not rely on the unknown limit. Therefore, we will introduce the following definition:

Definition: Cauchy Sequence

Let $\{a_j\}_{j=1}^{\infty}$ be a sequence of real (or complex) numbers. We say that the sequence satisfies the Cauchy criterion (or simply is Cauchy) if for each $\varepsilon > 0$ there is an integer N > 0 such that if j, k > N then

| aj - ak | < ε

This definition states precisely what it means for the elements of a sequence to get closer together, and to stay close together. Of course, we want to know what the relation between Cauchy sequences and convergent sequences is.

Theorem: Completeness Theorem in R

Let $\{a_j\}_{j=1}^{\infty}$ be a Cauchy sequence of real numbers. Then the sequence is bounded.

Let $\{a_j\}_{j=1}^{\infty}$ be a sequence of real numbers. The sequence is Cauchy if and only if it converges to some limit *a*.

Proof:

The proof of the first statement follows closely the proof of the corresponding result for convergent sequences. Can you do it ?

To prove the second, more important statement, we have to prove two parts:

First, assume that the sequence converges to some limit a. Take any $\varepsilon > 0$. There exists an integer N such that if $j > N$ then $| a_j - a | < \varepsilon /2$. Hence:

$| a_j - a_k | \leq | a_j - a | + | a - a_k| < 2 \varepsilon / 2 = \varepsilon$

if $j, k > N$. Thus, the sequence is Cauchy.

Second, assume that the sequence is Cauchy (this direction is much harder). Define the set

$\boldsymbol{S} = \{x \in \boldsymbol{R}: x < a_j \text{ for all } j \text{ except for finitely many}\}$

Since the sequence is bounded (by part one of the theorem), say by a constant M, we know that every term in the sequence is bigger than $-M$. Therefore $-M$ is contained in $\boldsymbol{S}$. Also, every term of the sequence is smaller than M, so that $\boldsymbol{S}$ is bounded by M. Hence, $\boldsymbol{S}$ is a non-empty, bounded subset of the real numbers, and by the least upper bound property it has a well-defined, unique least upper bound. Let

$a = sup(\boldsymbol{S})$

We will now show that this a is indeed the limit of the sequence. Take any $\varepsilon > 0$, and choose an integer $N > 0$ such that

$| a_j - a_k | < \varepsilon / 2$

if $j, k > N$. In particular, we have:

$| a_j - a_{N+1} | < \varepsilon / 2$

if $j > N$, or equivalently

$- \varepsilon / 2 < a_j - a_{N+1} < \varepsilon / 2$

Hence we have:

$a_j > a_{N+1} - \varepsilon / 2$

for $j > N$. Thus, $a_{N+1} - \varepsilon / 2$ is in the set $\boldsymbol{S}$, and we have that

$a \geq a_{N+1} - \varepsilon / 2$

It also follows that

$a_j < a_{N+1} + \varepsilon / 2$

for $j > N$. Thus, $a_{N+1} + \varepsilon / 2$ is not in the set $\boldsymbol{S}$, and therefore

$a \leq a_{N+1} + \varepsilon / 2$

But now, combining the last several line, we have that:

$|a - a_{N+1} | < \varepsilon / 2$

and together with the above that results in the following:

$| a - a_j | < |a - a_{N+1} | + | a_{N+1} - a_j | < 2 \varepsilon / 2 = \varepsilon$

for any $j > N$.

Thus, by considering Cauchy sequences instead of convergent sequences we do not need to refer to the unknown limit of a sequence, and in effect both concepts are the same.

Note that the Completeness Theorem not true if we consider only rational numbers. For example, the sequence 1, 1.4, 1.41, 1.414, ... (convergent to the square root of 2) is Cauchy, but does not converge to a rational number. Therefore, the rational numbers are not complete, in the sense that not every Cauchy sequence of rational numbers converges to a rational number.

Hence, the proof will have to use that property which distinguishes the reals from the rationals: the least upper bound property.

Subsequences

So far we have learned the basic definitions of a sequence (a function from the natural numbers to the Reals), the concept of convergence, and we have extended that concept to one which does not pre-suppose the unknown limit of a sequence (Cauchy sequence).

Unfortunately, however, not all sequences converge. We will now introduce some techniques for dealing with those sequences. The first is to change the sequence into a convergent one (extract subsequences) and the second is to modify our concept of limit (*lim sup* and *lim inf*).

Definition: Subsequence

Let $\{a_j\}_{j=1}^{\infty}$ be a sequence. When we extract from this sequence only certain elements and drop the remaining ones we obtain a new sequences consisting of an infinite subset of the original sequence. That sequence is called a subsequence and denoted by $\{a_{j_k}\}_{k=1}^{\infty}$

One can extract infinitely many subsequences from any given sequence.

Examples:

Take the sequence $\{(-1)^j\}_{j=1}^{\infty}$. Extract every other member, starting with the first. Then do the same, starting with the **second.**

The sequence in question is

$\{(-1)^j\}_{j=1}^{\infty}$ = *{-1, 1, -1, 1, -1, 1, ...}*

If we extract every second number, starting with the first, we get:

{-1, -1, -1, -1, ...}

This subsequence now converges to -1.

If we extract every second number, starting with the second, we get:

{1, 1, 1, 1, 1, ...}

Notes

This subsequence now converges to 1.

Take the sequence $\left\{ \frac{1}{j} \right\}_{j=1}^{\infty}$. Extract three different subsequences of your choice and look at the convergence behavior of these subsequences.

The sequence in question is:

$\left\{ \frac{1}{j} \right\}_{j=1}^{\infty}$ *= {1, 1/2, 1/3, 1/4, 1/5, 1/6, ... }*

which converges to zero. Now let us extract some subsequences:

Extracting the even terms yields the subsequence

{1/2, 1/4, 1/6, 1/8, 1/10, ...}

which converges to zero (prove it !).

Extracting the odd terms yields the subsequence

{1, 1/3, 1/5, 1/7, 1/9, ...}

which converges to zero (prove it !).

Extracting every third member yields the sequence

{1, 1/4, 1/7, 1/10, 1/13, ...}

which converges to zero (prove it !).

Hence, all three subsequences converge to zero. This is an illustration of a general result: if a sequence converges to a limit L then every subsequence extracted from it will also converge to that limit L.

The last example is an indication of a general result:

Proposition 3.3.3: Subsequences from Convergent Sequence

If $\left\{ a_j \right\}_{j=1}^{\infty}$ is a convergent sequence, then every subsequence of that sequence converges to the same limit

If $\left\{ a_j \right\}_{j=1}^{\infty}$ is a sequence such that every possible subsequence extracted from that sequences converge to the same limit, then the original sequence also converges to that limit.

Proof:

The first statement is easy to prove: Suppose the original sequence $\{a_j\}$ converges to some limit L. Take any sequence n_j of the natural numbers and consider the corresponding subsequence of the original sequence. For any $\varepsilon > 0$ there exists an integer N such that

$| a_n - L | < \varepsilon$

as long as $n > N$. But then we also have the same inequality for the subsequence as long as $n_j > N$. Therefore any subsequence must converge to the same limit L.

The second statement is just as easy. Suppose $\{a_j\}$ is a sequence such that every subsequence extracted from it converges to the same limit L. Now take any $\varepsilon > 0$. Extract from the original sequence every other element, starting with the first. The resulting subsequence converges to L by assumption, i.e. there exists an integer N such that

$| a_j - L | < \varepsilon$

where j is odd and $j > N$. Now extract every other element, starting with the second. The resulting subsequence again converges to L, so that

$| a_j - L | < \varepsilon$

where j is even and $j > N$. But now we take any j, even or odd, and assume that $j > N$

- if j is odd, then $| a_j - L | < \varepsilon$ because a_j is part of the first subsequence
- if j is even, then $| a_j - L | < \varepsilon$ because a_j is part of the second subsequence

Hence, the original sequence must also converge to L.

Note that we can see from the proof that if the "even" and "odd" subsequence of a sequence converge to the same limit L, then the full sequence must also converge to L. It is not enough to just say that the "even" and "odd" subsequence simply converge, they must converge to the *same* limit.

The next statement is probably one on the most fundamental results of basic real analysis, and generalizes the above proposition. It also explains why subsequences can be useful, even if the original sequence does not converge.

Theorem: Bolzano-Weierstrass

Let $\{a_j\}_{j=1}^{\infty}$ be a sequence of real numbers that is bounded. Then there exists a subsequence $\{a_{j_k}\}_{k=1}^{\infty}$ that converges.

Proof:

Since the sequence is bounded, there exists a number M such that $| a_j | < M$ for all j. Then:

either *[-M, 0]* or *[0, M]* contains infinitely many elements of the sequence

Say that *[0, M]* does. Choose one of them, and call it a_{j_1}

either *[0, M/2]* or *[M/2, M]* contains infinitely many elements of the (original) sequence.

Say it is *[0, M/2]*. Choose one of them, and call it a_{j_2}

either *[0, M/4]* or *[M/4, M/2]* contains infinitely many elements of the (original) sequence

This time, say it is *[M/4, M/2]*. Pick one of them and call it a_{j_3}

Keep on going in this way, halving each interval from the previous step at the next step, and choosing one element from that new interval. Here is what we get:

- $|a_{j_1} - a_{j_2}| < M$, because both are in *[0, M]*
- $|a_{j_2} - a_{j_3}| < M / 2$, because both are in *[0, M/2]*
- $|a_{j_3} - a_{j_4}| < M / 4$, because both are in *[M/2, M/4]*

and in general, we see that

$|a_{j_k} - a_{j_{k+1}}| < M / 2^{k-1}$

because both are in an interval of length $M / 2^{k-1}$. So, this proves that consecutive elements of this subsequence are close together. That is not enough, however, to say that the sequence is Cauchy, since for that not only consecutive elements must be close together, but all elements must get close to each other eventually.

So: take any $\varepsilon > 0$, and pick an integer N such that ???...??? (This trick is often used: first, do some calculation, then decide what the best choice for N should be. Right now, we have no way of knowing a good choice). Pretending, however, that we knew this choice of N, we continue the proof. For any k, $m > N$ (with $m > k$) we have:

$$|a_{j_k} - a_{j_m}| = |(a_{j_k} - a_{j_{k+1}}) + (a_{j_{k+1}} - a_{j_{k+2}}) + \ldots + (a_{j_{m-1}} - a_{j_m})|$$

$$\leq |a_{j_k} - a_{j_{k+1}}| + |a_{j_{k+1}} - a_{j_{k+2}}| + \ldots + |a_{j_{m-1}} - a_{j_m}|$$

$$= M\left(1/2^{k-1} + 1/2^{k} + 1/2^{k+1} \ldots + 1/2^{m-2} \right)$$

$$= M/2^{k-1}\left(1 + 1/2 + 1/2^{2} + \ldots + 1/2^{m-k-1} \right)$$

$$\leq M/2^{k-1}\left(1 + 1/2 + 1/2^{2} + 1/2^{3} \ldots \right)$$

$$= M/2^{k-1} \sum_{j=0}^{\infty} (1/2)^{j} = 2M/2^{k-1} = M/2^{k-2}$$

Now we can see the choice for N: we want to make is so large, such that whenever k, $m > N$, the difference between the members of the subsequence is less than the prescribed ε. What is therefore the right choice for N to finish the proof?

Example:

The sequence $\{\sin(j)\}_{j=1}^{\infty}$ does not converge, but we can extract a convergent subsequence.

Since $|\sin(x)| < 4$, the sequence is clearly bounded above and below (the sequence is also, of course, bounded by 1).

Therefore, using the Bolzano-Weierstrass theorem, there exists a convergent subsequence.

However, it is nearly impossible to actually list this subsequence. The Bolzano-Weierstrass theorem does guaranty the existence of that subsequence, but says nothing about how to obtain it.

The original sequence *{ sin(j) }*, incidentally, does not converge. The proof of this is not so easy, but if we assume that the second part of this example has been proved, it would be easy. Remember that the second part of this example states that given *any* number *L* with $|L| < 1$ there exists a subsequence of $\{\sin(j)\}_{j=1}^{\infty}$ that converges to *L*. If that was true the original sequence cannot converge, because otherwise all its subsequences would have to converge to the same limit.

Of course this proof is only valid if this - more complicated - statement can be proved.

5.6 CANTOR'S INTERSECTION THEOREM

Cantor's intersection theorem refers to two closely related theorems in general topology and real analysis, named after Georg Cantor, about intersections of decreasing nested sequences of non-empty compact sets.

Statement for Real Numbers

The theorem in real analysis draws the same conclusion for closed and bounded subsets of the set of real numbers ℝ. It states that a decreasing nested sequence (C_k) of non-empty, closed and bounded subsets of ℝ has a non-empty intersection.

This version follows from the general topological statement in light of the Heine–Borel theorem, which states that sets of real numbers are compact if and only if they are closed and bounded. However, it is typically used as a lemma in proving said theorem, and therefore warrants a separate proof.

As an example, if $C_k = [0, 1/k]$, the intersection over C_k is {0}. On the other hand, both the sequence of open bounded sets $C_k = [0, 1/k]$ and the sequence of unbounded closed sets $C_k = [k, \infty)$ have empty intersection. All these sequences are properly nested.

This version of the theorem generalizes to ℝ, the set of n-element vectors of real numbers $\mathbb{R}^n$, but does not generalize to arbitrary metric spaces. For example, in the space of rational numbers, the sets

$$C_k = [\sqrt{2}, \sqrt{2} + 1/k] = (\sqrt{2}, \sqrt{2} + 1/k)$$

are closed and bounded, but their intersection is empty.

Note that this contradicts neither the topological statement, as the sets C_k are not compact, nor the variant below, as the rational numbers are not complete with respect to the usual metric.

A simple corollary of the theorem is that the Cantor set is nonempty, since it is defined as the intersection of a decreasing nested sequence of sets, each of which is defined as the union of a finite number of closed intervals; hence each of these sets is non-empty, closed, and bounded. In fact, the Cantor set contains uncountably many points.

Theorem. Let C_k be a family of non-empty, closed, and bounded subsets of ℝ satisfying

$$C_0 \supset C_1 \supset \cdots C_n \supset C_{n+1} \cdots .$$

Then,

$$\left(\bigcap_k C_k\right) \neq \emptyset.$$

Each nonempty, closed, and bounded subset $C_k \subset \mathbf{R}$ admits a minimal element x_k. Since for each k, we have

$$x_{k+1} \in C_{k+1} \subseteq C_k,$$

it follows that

$$x_k \leq x_{k+1},$$

so x_k is an increasing sequence contained in the bounded set C_0. The monotone convergence theorem for bounded sequences of real numbers now guarantees the existence of a limit point

$$x = \lim_{k\to\infty} x_k.$$

For fixed *k*, $x_j \in C_k$ for all $j \geq k$ and since C_k was closed and x is limit point it follows that $x \in C_k$. Our choice of k was arbitrary, hence x belongs to $\bigcap_k C_k$ and the proof is complete.

Variant in complete metric spaces

In a complete metric space, the following variant of Cantor's intersection theorem holds.

Theorem: Suppose that X is a complete metric space, and C_k is a sequence of non-empty closed nested subsets of X whose diameters tend to zero:

$$\lim_{k\to\infty} \operatorname{diam}(C_k) = 0,$$

where $\operatorname{diam}(C_k)$ *is defined by*

$$\operatorname{diam}(C_k) = \sup\{d(x,y) | x, y \in C_k\}.$$

Then the intersection of the C_k *contains exactly one point:*

$$\bigcap_{k=1}^{\infty} C_k = \{x\}$$

for some x in X.

A proof goes as follows. Since the diameters tend to zero, the diameter of the intersection of the C_k is zero, so it is either empty or consists of a single point. So it is sufficient to show that it is not empty. Pick an element $x_k \in C_k$ for each k. Since the diameter of C_k tends to zero and the C_k are nested, x_k the form a Cauchy sequence. Since the metric space is complete this Cauchy sequence converges to some point x. Since each C_k is closed, and x is a limit of a sequence in C_k, x must lie in C_k. This is true for every k, and therefore the intersection of the C_k must contain x.

A converse to this theorem is also true: if X is a metric space with the property that the intersection of any nested family of non-empty closed subsets whose diameters tend to zero is non-empty, then X is a complete metric space.

5.7 BAIRE CATEGORY THEOREM

The Baire category theorem (BCT) is an important result in general topology and functional analysis. The theorem has two forms, each of which gives sufficient conditions for a topological space to be a Baire space

Statement of the theorem

A Baire space is a topological space with the following property: for each countable collection of open dense sets $\{U_n\}_{n=1}^{\infty}$, their intersection $\bigcap_{n=1}^{\infty} U_n$ is dense.

(BCT1) Every complete metric space is a Baire space. Thus every completely metrizable topological space is a Baire space. More generally, every topological space that is homeomorphic to an open subset of a complete pseudometric space is a Baire space.

(BCT2) Every locally compact Hausdorff space is a Baire space. The proof is similar to the preceding statement; the finite intersection property takes the role played by completeness.

Neither of these statements directly implies the other, since there are complete metric spaces that are not locally compact, and there are locally compact Hausdorff spaces that are not metrizable .

(BCT3) A non-empty complete metric space, or any of its subsets with nonempty interior, is not the countable union of nowhere-dense sets.

Relation to the axiom of choice

The proof of BCT1 for arbitrary complete metric spaces requires some form of the axiom of choice; and in fact BCT1 is equivalent over ZF to a weak form of the axiom of choice called the axiom of dependent choices.

A restricted form of the Baire category theorem, in which the complete metric space is also assumed to be separable, is provable in ZF with no additional choice principles. This restricted form applies in particular to the real line, the Baire space ωω, the Cantor space 2ω, and a separable Hilbert space such as L2(Rn)

Uses of the theorem

BCT1 is used in functional analysis to prove the open mapping theorem, the closed graph theorem and the uniform boundedness principle.

BCT1 also shows that every complete metric space with no isolated points is uncountable. (If X is a countable complete metric space with no isolated points, then each singleton {x} in X is nowhere dense, and so X is of first category in itself.) In particular, this proves that the set of all real numbers is uncountable.

BCT1 shows that each of the following is a Baire space:

- The space ℝ of real numbers
- The irrational numbers, with the metric defined by $d(x,y)=\frac{1}{n+1}$ where n is the first index for which the continued fraction expansions of x and y differ (this is a complete metric space)
- The Cantor set

By BCT2, every finite-dimensional Hausdorff manifold is a Baire space, since it is locally compact and Hausdorff. This is so even for non-paracompact (hence nonmetrizable) manifolds such as the long line.

BCT is used to prove Hartogs's theorem, a fundamental result in the theory of several complex variables.

Proof:

The following is a standard proof that a complete pseudometric space X is a Baire space.

Let U_n be a countable collection of open dense subsets. We want to show that the intersection $\bigcap U_n$ A subset is dense if and only if every nonempty open subset intersects it. Thus, to show that the intersection is dense, it is sufficient to show that any nonempty open set **W** in **X** has a point **x** in common with all of the U_n. Since **U_1** is dense, **W** intersects **U_1**; thus, there is a point x_1 and $0 < r_1 < 1$ such that:

$\overline{B}(x_1, r_1) \subseteq W \cap U_1$

denote an open and closed ball, respectively, centered at x with radius r.

U_n is dense, we can continue recursively to find a pair of sequences

x_n and $0 < r_n < \frac{1}{n}$ such that:

$\overline{B}(x_n, r_n) \subset B(x_{n-1}, r_{n-1}) \cap U_n$

Since $x_n \in B(x_m, r_m)$ when $n > m$, we have that X_n is Cauchy, and hence X_n converges to some limit x by completeness. For any n

$x \in \overline{B}(x_n, r_n)$.

Therefore, $x \in W$ and $x \in U_n$ for all n.

5.8LET SUM UP

In this unit, we have covered the following points:

We defined a sequence in a metric space (X, d) and discussed its Convergence. We defined subsequence’s of a sequence and have shown the relationship between convergence of a sequence and its subsequence’s.

We defined Cauchy sequences and explained the connection between Cauchy sequences & convergence. A Cauchy sequence is convergent if and only if it has a convergent subsequence.

We discussed two important theorems and explained the importance of them.1) Cantor's Intersection Theorem 2) Baire Category Theorem

5.9 KEYWORD

Cauchy Sequence

Sub Sequence

Baire Category

5.10 QUESTIONS FOR REVIEW

Q. 1 State and prove Baire’s Category Theorem.

Q. 2 State and prove Cantor's Intersection Theorem.

Q. 3 Prove with example subsequence of a sequence.

Q. 4 State and prove Squeez's theorem.

5.11 SUGGESTION READING & REFERENCE

- D'Angelo, J. P. and West, D. B. *Mathematical Thinking: Problem-Solving and Proofs, 2nd ed.* Upper Saddle River, NJ: Prentice-Hall, 2000.
- Jeffreys, H. and Jeffreys, B. S. "Bounded, Unbounded, Convergent, Oscillatory." §1.041 in *Methods of Mathematical Physics, 3rd ed.* Cambridge, England: Cambridge University Press, pp. 11-12, 1988.
- Courant, Richard (1961). "Differential and Integral Calculus Volume I", Blackie & Son, Ltd., Glasgow.
- Frank Morley and James Harkness A treatise on the theory of functions (New York: Macmillan, 1893)

5.12 ANSWERS TO CHECK YOUR PROGRESS

Check in progress -1

1 Check section 5.3

2 Check section 5.2

UNIT - 6 : LEBESGUE MEASURE

STRUCTURE

6.1 INTRODUCTION

In measure theory, the **Lebesgue measure**, named after French mathematician Henri Lebesgue, is the standard way of assigning a measure to subsets of *n*-dimensional Euclidean space. For n = 1, 2, or 3, it coincides with the standard measure of length, area, or volume. In general, it is also called ***n*-dimensional volume**, ***n*-**

volume, or simply **volume**. It is used throughout real analysis, in particular to define Lebesgue integration. Sets that can be assigned a Lebesgue measure are called **Lebesgue-measurable**; the measure of the Lebesgue-measurable set A is here denoted by $\lambda(A)$.

Henri Lebesgue described this measure in the year 1901, followed the next year by his description of the Lebesgue integral. Both were published as part of his dissertation in 1902.

The Lebesgue measure is often denoted by dx, but this should not be confused with the distinct notion of a volume form.

Definition

Given a subset $E \subseteq \mathbb{R}$, with the length of interval $I = [a, b]$ (or $I = (a, b)$) given by $\ell(I)\ b - a$, the Lebesgue outer measure $\lambda^*(E)$ is defined as

$$\lambda^*(E) = \inf\left\{\sum_{k=1}^{\infty} \ell(I_k) : (I_k)_{k\in\mathbb{N}} \text{ is a sequence of intervals with open boundaries with } E \subseteq \bigcup_{k=1}^{\infty} I_k\right\}.$$

The Lebesgue measure is defined on the Lebesgue σ-algebra, which is the collection of all sets E which satisfy the "Carathéodory criterion" which requires that for every $A \subseteq \mathbb{R}$,

$$\lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \cap E^c)$$

For any set in the Lebesgue σ-algebra, its Lebesgue measure is given by its Lebesgue outer measure $\lambda(E) = \lambda^*(E)$.

Sets that are not included in the Lebesgue σ-algebra are not Lebesgue-measurable. Such sets do exist (e.g. Vitali sets), i.e., the Lebesgue σ-algebra is strictly contained in the power set of $\mathbb{R}$.

Intuition

The first part of the definition states that the subset E of the real numbers is reduced to its outer measure by coverage by sets of open intervals. Each of these sets of intervals I covers E in the sense that when the intervals are combined together by union, they contain E. The total length of any covering interval set can easily overestimate the measure of E, because E is a subset of the union of the intervals, and so the intervals may include points which are not in E. The Lebesgue outer measure emerges as the greatest lower bound (infimum) of the lengths from among all possible such sets. Intuitively, it is the total length of those interval sets which fit E most tightly and do not overlap.

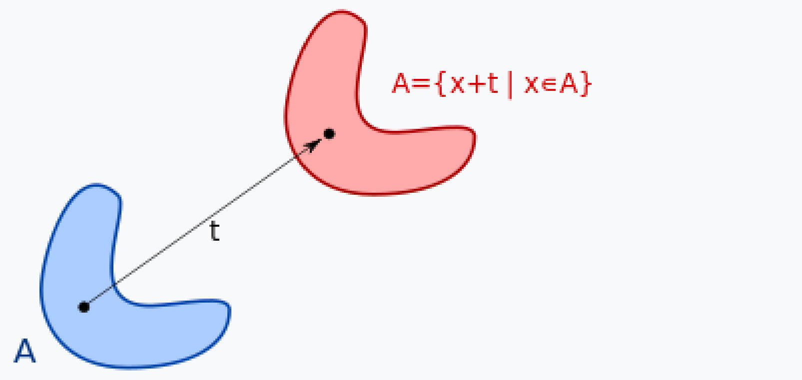
That characterizes the Lebesgue outer measure. Whether this outer measure translates to the Lebesgue measure proper depends on an additional condition. This condition is tested by taking subsets A of the real numbers using E as an instrument to split A into two partitions: the part of A which intersects with E and the remaining part of A which is not in E: the set difference of A and E. These partitions of A are subject to the outer measure. If for all possible such subsets A of the real numbers, the partitions of A cut apart by E have outer measures whose sum is the outer measure of A, then the outer Lebesgue measure of E gives its Lebesgue measure. Intuitively, this condition means that the set E must not have some curious properties which causes a discrepancy in the measure of another set when E is used as a "mask" to "clip" that set, hinting at the existence of sets for which the Lebesgue outer measure does not give the Lebesgue measure. (Such sets are, in fact, not Lebesgue-measurable.)

Examples

- Any open or closed interval $[a, b]$ of real numbers is Lebesgue-measurable, and its Lebesgue measure is the length $b - a$. The open interval (a, b) has the same measure, since the difference between the two sets consists only of the end points a and b and has measure zero.
- Any Cartesian product of intervals $[a, b]$ and $[c, d]$ is Lebesgue-measurable, and its Lebesgue measure is $(b - a)(d - c)$, the area of the corresponding rectangle.
- Moreover, every Borel set is Lebesgue-measurable. However, there are Lebesgue-measurable sets which are not Borel sets.
- Any countable set of real numbers has Lebesgue measure 0.
- In particular, the Lebesgue measure of the set of rational numbers is 0, although the set is dense in **R**.
- The Cantor set is an example of an uncountable set that has Lebesgue measure zero.

- If the axiom of determinacy holds then all sets of reals are Lebesgue-measurable. Determinacy is however not compatible with the axiom of choice.
- Vitali sets are examples of sets that are not measurable with respect to the Lebesgue measure. Their existence relies on the axiom of choice.
- Osgood curves are simple Plane curves with positive Lebesgue measure. The dragon curve is another unusual example.
- Any line in $\mathbb{R}^n$, for $n \geq 2$, has a zero Lebesgue measure. In general, every proper hyperplane has a zero Lebesgue measure in its ambient space.

6.2 LEBESGUE MEASURE PROPERTIES



Translation invariance: The Lebesgue measure of and are the same.

The Lebesgue measure on $\mathbf{R}^n$ has the following properties:

1. If A is a cartesian product of intervals $I_1 \times I_2 \times \ldots \times I_n$, then A isLebesgue-measurable & $\lambda(A) = |I_1| \cdot |I_2| \cdots |I_n|$. Here, $|I|$ denotes the length of the interval I.
2. If A is a disjoint union of countably many disjoint Lebesgue-measurable sets, then A is itself Lebesgue-measurable and $\lambda(A)$ is equal to the sum (or infinite series) of the measures of the involved measurable sets.
3. If A is Lebesgue-measurable, then so is its complement.
4. $\lambda(A) \geq 0$ for every Lebesgue-measurable set A.

5. If A and B are Lebesgue-measurable and A is a subset of B, then $\lambda(A) \leq \lambda(B)$. (A consequence of 2, 3 and 4.)
6. Countable unions and intersections of Lebesgue-measurable sets are Lebesgue-measurable. (Not a consequence of 2 and 3, because a family of sets that is closed under complements and disjoint countable unions need not be closed under countable unions: $\{\emptyset, \{1,2,3,4\}, \{1,2\}, \{3,4\}, \{1,3\}, \{2,4\}\}$.)
7. If A is an open or closed subset of $\mathbf{R}^n$ (or even Borel set, see metric space), then A is Lebesgue-measurable.
8. If A is a Lebesgue-measurable set, then it is "approximately open" and "approximately closed" in the sense of Lebesgue measure (see the regularity theorem for Lebesgue measure).
9. A Lebesgue-measurable set can be "squeezed" between a containing open set and a contained closed set. This property has been used as an alternative definition of Lebesgue measurability. More precisely $E \subset \mathbb{R}$, is Lebesgue-measurable if and only if for every $\epsilon>0$ there exist an open set G and a closed set F such that $F \subset E \subset G$ and $\lambda(G \setminus F) < \varepsilon$.
10. A Lebesgue-measurable set can be "squeezed" between a containing G_δ set and a contained F_σ. I.e, if A is Lebesgue-measurable then there exist a G_δ set G and an F_σ F such that $G \supseteq A \supseteq F$ and $\lambda(G \setminus A) = \lambda(A \setminus F) = 0$.
11. Lebesgue measure is both locally finite and inner regular, and so it is a Radon measure.
12. Lebesgue measure is strictly positive on non-empty open sets, and so its support is the whole of $\mathbf{R}^n$.
13. If A is a Lebesgue-measurable set with $\lambda(A) = 0$ *(a null set), then every subset of* A *is also a null set. A fortiori, every subset of* A *is measurable.*
14. If A is Lebesgue-measurable and x is an element of $\mathbf{R}^n$, then the *translation of* A *by* x, defined by $A + x =$

$\{a + x : a \in A\}$, is also Lebesgue-measurable and has the same measure as A.

15. If A is Lebesgue-measurable $\delta > 0$ and , then the *dilation of A by* δ defined by $\delta A = \{\delta x : x \epsilon A\}$ is also Lebesgue-measurable and has measure $\delta^n \lambda(A)$.
16. More generally, if T is a linear transformation and A is a measurable subset of $\mathbf{R}^n$, then $T(A)$ is also Lebesgue-measurable and has the measure $|\det(T)| \lambda(A)$.

All the above may be succinctly summarized as follows:

The Lebesgue-measurable sets form a σ-algebra containing all products of intervals, and λ is the unique complete translation-invariant measure on that σ-algebra with $\lambda([0,1] \times [0,1] \times \cdots \times [0,1]) = 1$.

The Lebesgue measure also has the property of being σ-finite.

6.3 NULL SETS

A subset of $\mathbf{R}^n$ is a *null set* if, for every ε > 0, it can be covered with countably many products of n intervals whose total volume is at most ε. All countable sets are null sets.

If a subset of $\mathbf{R}^n$ has Hausdorff dimension less than n then it is a null set with respect to n-dimensional Lebesgue measure. Here Hausdorff dimension is relative to the Euclidean metric on $\mathbf{R}^n$ (or any metric Lipschitz equivalent to it). On the other hand, a set may have topological dimension less than n and have positive n-dimensional Lebesgue measure. An example of this is the Smith–Volterra–Cantor set which has topological dimension 0 yet has positive 1-dimensional Lebesgue measure.

In order to show that a given set A is Lebesgue-measurable, one usually tries to find a "nicer" set B which differs from A only by a null set (in the sense that the symmetric difference $(A - B) \cup (B - A)$ is a null set) and then show that B can be generated using countable unions and intersections from open or closed sets.

Check your Progress -1

Q. 1 What is the defination of Lebesgue measure ?

…………………………………………………………………………

…………………………………………………………………………

…………………………………………………………………………

…………………………………………………………………………

Q.2 Explain Null sets.

…………………………………………………………………………

…………………………………………………………………………

…………………………………………………………………………

…………………………………………………………………………

6.4 CONSTRUCTION OF THE LEBESGUE MEASURE

The modern construction of the Lebesgue measure is an application of Carathéodory's extension theorem. It proceeds as follows.

Fix $n \in \mathbf{N}$. A **box** in $\mathbf{R}^n$ is a set of the form

$$B = \prod_{i=1}^{n} [a_i, b_i] ,$$

where $b_i \geq a_i$, and the product symbol here represents a Cartesian product. The volume of this box is defined to be

$$\operatorname{vol}(B) = \prod_{i=1}^{n} (b_i - a_i) .$$

For *any* subset A of $\mathbf{R}^n$, we can define its outer measure $\lambda^*(A)$ by:

$$\lambda^*(A) = \inf \left\{ \sum_{B \in \mathcal{C}} \operatorname{vol}(B) : \mathcal{C} \text{ is a countable collection of boxes whose union covers } A \right\} .$$

We then define the set A to be Lebesgue-measurable if for every subset S of $\mathbf{R}^n$,

$$\lambda^*(S) = \lambda^*(S \cap A) + \lambda^*(S \setminus A) .$$

These Lebesgue-measurable sets form a σ-algebra, and the Lebesgue measure is defined by $\lambda(A) = \lambda^*(A)$ for any Lebesgue-measurable set A.

The existence of sets that are not Lebesgue-measurable is a consequence of a certain set-theoretical axiom, the axiom of choice, which is independent from many of the conventional systems of axioms for set theory. The Vitali theorem, which follows from the axiom, states that there exist subsets of **R** that are not Lebesgue-measurable. Assuming the axiom of choice, non-measurable sets with many surprising properties have been demonstrated, such as those of the Banach–Tarski paradox.

In 1970, Robert M. Solovay showed that the existence of sets that are not Lebesgue-measurable is not provable within the framework of Zermelo–Fraenkel set theory in the absence of the axiom of choice.

6.4.1Relation To Other Measures

The Borel measure agrees with the Lebesgue measure on those sets for which it is defined; however, there are many more Lebesgue-measurable sets than there are Borel measurable sets. The Borel measure is translation-invariant, but not complete.

The Haar measure can be defined on any locally compact group and is a generalization of the Lebesgue measure ($\mathbf{R}^n$ with addition is a locally compact group).

The Hausdorff measure is a generalization of the Lebesgue measure that is useful for measuring the subsets of $\mathbf{R}^n$ of lower dimensions than n, like submanifolds, for example, surfaces or curves in $\mathbf{R}^3$ and fractal sets. The Hausdorff measure is not to be confused with the notion of Hausdorff dimension.

It can be shown that there is no infinite-dimensional analogue of Lebesgue measure.

6.4.2 Lebesgue's Density Theorem

Lebesgue's density theorem states that for any Lebesgue measurable set , the "density" of A is 0 or 1 at almost every point in . Additionally, the "density" of A is 1 at almost every point in A. Intuitively, this means that the "edge" of A, the set of points in A whose "neighborhood" is partially in A and partially outside of A, is negligible.

Let μ be the Lebesgue measure on the Euclidean space $\mathbf{R}^n$ and A be a Lebesgue measurable su bset of $\mathbf{R}^n$. Define the **approximate density** of A in a ε-neighborhood of a point x in $\mathbf{R}^n$ as

$$d_\varepsilon(x) = \frac{\mu(A \cap B_\varepsilon(x))}{\mu(B_\varepsilon(x))}$$

where B_ε denotes the closed ball of radius ε centered at x.

Lebesgue's density theorem asserts that for almost every point x of A the **density**

$$d(x) = \lim_{\varepsilon \to 0} d_\varepsilon(x)$$

exists and is equal to 1.

In other words, for every measurable set A, the density of A is 0 or 1 almost everywhere in $\mathbf{R}^n$.[1] However, it is a curious fact that if $\mu(A) > 0$ and $\mu(\mathbf{R}^n \setminus A) > 0$, then there are always points of $\mathbf{R}^n$ where the density is neither 0 nor 1.

For example, given a square in the plane, the density at every point inside the square is 1, on the edges is 1/2, and at the corners is 1/4. The set of points in the plane at which the density is neither 0 nor 1 is non-empty (the square boundary), but it is negligible.

The Lebesgue density theorem is a particular case of the Lebesgue differentiation theorem.

Thus, this theorem is also true for every finite Borel measure on $\mathbf{R}^n$ instead of Lebesgue measure, see Discussion.

6.4.3Liouville Number

In number theory, a **Liouville number** is a real number x with the property that, for every positive integer n, there exist infinitely many pairs of integers (p, q) with $q > 1$ such that

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^n}.$$

Liouville numbers are "almost rational", and can thus be approximated "quite closely" by sequences of rational numbers. They are precisely the transcendental numbers that can be more closely approximated by

Notes

rational numbers than any algebraic irrational number. In 1844, Joseph Liouville showed that all Liouville numbers are transcendental, thus establishing the existence of transcendental numbers for the first time.

6.4.4 The existence of Liouville numbers (Liouville's constant)

Here we show that Liouville numbers exist by exhibiting a construction that produces such numbers.

For any integer $b \geq 2$, and any sequence of integers $(a_1, a_2, \ldots,)$, such that $a_k \in \{0, 1, 2, \ldots, b-1\}$ for all $k \in \{1, 2, 3, \ldots\}$ and there are infinitely many k with $a_k \neq 0$, define the number

$$x = \sum_{k=1}^{\infty} \frac{a_k}{b^{k!}}$$

In the special case when $b = 10$, and $a_k = 1$, for all k, the resulting number x is called **Liouville's constant:**

$L =$

0.**11**000**1**00000000000000000**1**000000000000000000000000000000000
00000000000000000000000000000000000000000000000000000000000
000**1**...

It follows from the definition of x that its base-b representation is

$$x = (0.a_1a_2000a_3 00000000000000000a_4 00000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000a_5\ldots)_b$$

where the nth term is separated from the next term by $(nn! - 1)$ zeros.

Since this base-b representation is non-repeating it follows that x cannot be rational. Therefore, for any rational number p/q, we have
$|x - p/q\,| > 0$.

Now, for any integer $n \geq 1$, define q_n and p_n as follows:

$$q_n = b^{n!}\,; \quad p_n = q_n \sum_{k=1}^{n} \frac{a_k}{b^{k!}} = \sum_{k=1}^{n} a_k b^{n!-k!}\,.$$

Then

$$0 < \left|x - \frac{p_n}{q_n}\right| = \left|x - \frac{q_n \sum_{k=1}^{n} \frac{a_k}{b^{k!}}}{q_n}\right| = \left|x - \sum_{k=1}^{n} \frac{a_k}{b^{k!}}\right| = \left|\sum_{k=1}^{\infty} \frac{a_k}{b^{k!}} - \sum_{k=1}^{n} \frac{a_k}{b^{k!}}\right| = \left|\left(\sum_{k=1}^{n} \frac{a_k}{b^{k!}} + \sum_{k=n+1}^{\infty} \frac{a_k}{b^{k!}}\right) - \sum_{k=1}^{n} \frac{a_k}{b^{k!}}\right| = \sum_{k=n+1}^{\infty} \frac{a_k}{b^{k!}}$$

$$\leq \sum_{k=n+1}^{\infty} \frac{b-1}{b^{k!}} < \sum_{k=(n+1)!}^{\infty} \frac{b-1}{b^{k}} = \frac{b-1}{b^{(n+1)!}} + \frac{b-1}{b^{(n+1)!+1}} + \frac{b-1}{b^{(n+1)!+2}} + \ldots = \frac{b-1}{b^{(n+1)!}b^0} + \frac{b-1}{b^{(n+1)!}b^1} + \frac{b-1}{b^{(n+1)!}b^2} + \ldots = \frac{b-1}{b^{(n+1)!}} \sum_{k=0}^{\infty} \frac{1}{b^k}$$

$$= \frac{b-1}{b^{(n+1)!}} \cdot \frac{b}{b-1} = \frac{b}{b^{(n+1)!}} \leq \frac{b^{n!}}{b^{(n+1)!}} = \frac{1}{b^{(n+1)!-n!}} = \frac{1}{b^{(n+1)n!-n!}} = \frac{1}{b^{n(n!)+n!-n!}} = \frac{1}{b^{(n!)n}} = \frac{1}{{q_n}^n}$$

Therefore, we conclude that any such *x* is a Liouville number.

Notes on the proof

1. The inequality $\sum_{k=n+1}^{\infty} \frac{a_k}{b^{k!}} \leq \sum_{k=n}^{\infty} \frac{b-1}{b^{k!}}$ follows from the fact that "il existe" k, $a_k \in \{0, 1, 2, \ldots, b-1\}$. Therefore, at most, $a_k = b-1$. The largest possible sum would occur if the sequence of integers, $(a_1, a_2, \ldots)$, were (b-1, b-1, ...) where $a_k = b-1$, for all k. $\sum_{k=n+1}^{\infty} \frac{a_k}{b^{k!}}$, will thus be less than, or equal to, this largest possible sum.

2. The strong inequality $\sum_{k=n+1}^{\infty} \frac{b-1}{b^{k!}} < \sum_{k=(n+1)!}^{\infty} \frac{b-1}{b^{k}}$ follows from our motivation to eliminate the series by way of reducing it to a series for which we know a formula. In the proof so far, the purpose for introducing the inequality in 1. comes from intuition that $\sum_{k=0}^{\infty} \frac{1}{b^k} = \frac{b}{b-1}$ (the geometric series formula); therefore, if we can find an inequality from $\sum_{k=n+1}^{\infty} \frac{a_k}{b^{k!}}$ that introduces a series with (b-1) in the numerator, and if we can work to further reduce the denominator term $b^{k!}$ to b^k, as well as shifting the series indices from 0 to ∞, then we will be able to eliminate both series and (b-1) terms, getting us closer to a fraction of the form $\frac{1}{b^{(exponent)*n}}$, which is the end-goal of the proof. We further this motivation here by selecting now from the sum $\sum_{k=n+1}^{\infty} \frac{b-1}{b^{k!}}$ a partial sum. Observe that, for any term in $\sum_{k=n+1}^{\infty} \frac{b-1}{b^{k!}}$, since $b \geq 2$, then $\frac{b-1}{b^{k!}} < \frac{b-1}{b^k}$, for all k (except for when n=1). Therefore, $\sum_{k=n+1}^{\infty} \frac{b-1}{b^{k!}} < \sum_{k=n+1}^{\infty} \frac{b-1}{b^{k}}$ (since, even if n=1, all subsequent

Notes

terms are smaller). In order to manipulate the indices so that k starts at 0, we select a partial sum from within $\sum_{k=n+1}^{\infty} \frac{b-1}{b^k}$ (also less than the total value since it's a partial sum from a series whose terms are all positive). We will choose the partial sum formed by starting at k = (n+1)! which follows from our motivation to write a new series with k=0, namely by noticing that $b^{(n+1)!} = b^{(n+1)!}b^0$.

3. For the final inequality $\frac{b}{b^{(n+1)!}} \leq \frac{b^{n!}}{b^{(n+1)!}}$, we have chosen this particular inequality (true because $b \geq 2$, where equality follows if and only if n=1) because we wish to manipulate $\frac{b}{b^{(n+1)!}}$ into something of the form $\frac{1}{b^{(exponent)*n}}$. This particular inequality allows us to eliminate (n+1)! and the numerator, using the property that (n+1)! - n! = (n!)n, thus putting the denominator in ideal form for the substitution $q_n = b^{n!}$.

6.4.5 IRRATIONALITY

Here we will show that the number $x = c/d$, where c and d are integers and $d > 0$, cannot satisfy the inequalities that define a Liouville number. Since every rational number can be represented as such c/d, we will have proven that **no Liouville number can be rational**.

More specifically, we show that for any positive integer n large enough that $2^{n-1} > d > 0$ (that is, for any integer $n > 1 + \log_2(d)$) no pair of integers (p, q) exists that simultaneously satisfies the two inequalities

$$0 < \left|x - \frac{p}{q}\right| < \frac{1}{q^n}.$$

From this the claimed conclusion follows.

Let p and q be any integers with $q > 1$. Then we have,

$$\left|x - \frac{p}{q}\right| = \left|\frac{c}{d} - \frac{p}{q}\right| = \frac{|cq - dp|}{dq}$$

If $|cq - dp| = 0$, we would have

$$\left|x - \frac{p}{q}\right| = \frac{|cq - dp|}{dq} = 0,$$

meaning that such pair of integers (p, q) would violate the *first* inequality in the definition of a Liouville number, irrespective of any choice of n.

If, on the other hand, $|cq - dp| > 0$, then, since $cq - dp$ is an integer, we can assert the sharper inequality $|cq - dp| \geq 1$. From this it follows that

$$\left|x - \frac{p}{q}\right| = \frac{|cq - dp|}{dq} \geq \frac{1}{dq}$$

Now for any integer $n > 1 + \log_2(d)$, the last inequality above implies

$$\left|x - \frac{p}{q}\right| \geq \frac{1}{dq} > \frac{1}{2^{n-1}q} \geq \frac{1}{q^n}.$$

Therefore, in the case $|cq - dp| > 0$ such pair of integers (p, q) would violate the *second* inequality in the definition of a Liouville number, for some positive integer n.

We conclude that there is no pair of integers (p, q), with $q > 1$, that would qualify such an $x = c/d$ as a Liouville number.

Hence a Liouville number, if it exists, cannot be rational.

6.4.6 UNCOUNTABILITY

Consider, for example, the number

3.14000100000000000000000050000....

3.14(3 zeros)1(17 zeros)5(95 zeros)9(599 zeros)2(4319 zeros)6...

where the digits are zero except in positions $n!$ where the digit equals the nth digit following the decimal point in the decimal expansion of π.

As shown in the section on the existence of Liouville numbers, this number, as well as any other non-terminating decimal with its non-zero digits similarly situated, satisfies the definition of a Liouville number. Since the set of all sequences of non-null digits has the cardinality of the continuum, the same thing occurs with the set of all Liouville numbers.

Moreover, the Liouville numbers form a dense subset of the set of real numbers.

6.4.7 Liouville Numbers And Measure

From the point of view of measure theory, the set of all Liouville numbers *L* is small. More precisely, its Lebesgue measure is zero. The proof given follows some ideas by John C. Oxtoby.

For positive integers *n* > 2 and *q* ≥ 2 set:

$$V_{n,q} = \bigcup_{p=-\infty}^{\infty} \left(\frac{p}{q} - \frac{1}{q^n}, \frac{p}{q} + \frac{1}{q^n} \right)$$

we have

$$L \subseteq \bigcup_{q=2}^{\infty} V_{n,q}.$$

Observe that for each positive integer *n* ≥ 2 and *m* ≥ 1, we also have

$$L \cap (-m, m) \subseteq \bigcup_{q=2}^{\infty} V_{n,q} \cap (-m, m) \subseteq \bigcup_{q=2}^{\infty} \bigcup_{p=-mq}^{mq} \left(\frac{p}{q} - \frac{1}{q^n}, \frac{p}{q} + \frac{1}{q^n} \right).$$

Since

$$\left| \left(\frac{p}{q} + \frac{1}{q^n} \right) - \left(\frac{p}{q} - \frac{1}{q^n} \right) \right| = \frac{2}{q^n}$$

and *n* > 2 we have

$$m(L \cap (-m, m)) \leq \sum_{q=2}^{\infty} \sum_{p=-mq}^{mq} \frac{2}{q^n} = \sum_{q=2}^{\infty} \frac{2(2mq+1)}{q^n}$$

$$\leq (4m+1) \sum_{q=2}^{\infty} \frac{1}{q^{n-1}} \leq (4m+1) \int_1^{\infty} \frac{dq}{q^{n-1}} \leq \frac{4m+1}{n-2}.$$

Now

$$\lim_{n \to \infty} \frac{4m+1}{n-2} = 0$$

and it follows that for each positive integer *m*, $L \cap (-m, m)$ has Lebesgue measure zero. Consequently, so has *L*.

In contrast, the Lebesgue measure of the set *T* of *all* real transcendental numbers is infinite (since *T* is the complement of a null set).

In fact, the Hausdorff dimension of *L* is zero, which implies that the Hausdorff measure of *L* is zero for all dimension *d* > 0. Hausdorff dimension of *L* under other dimension functions has also been investigated.

STRUCTURE OF THE SET OF LIOUVILLE NUMBERS

For each positive integer *n*, set

$$U_n = \bigcup_{q=2}^{\infty} \bigcup_{p=-\infty}^{\infty} \left\{ x \in \mathbf{R} : 0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^n} \right\} = \bigcup_{q=2}^{\infty} \bigcup_{p=-\infty}^{\infty} \left(\frac{p}{q} - \frac{1}{q^n}, \frac{p}{q} + \frac{1}{q^n} \right) \setminus \left\{ \frac{p}{q} \right\}$$

The set of all Liouville numbers can thus be written as

$$L = \bigcap_{n=1}^{\infty} U_n = \bigcap_{n \in \mathbb{Z}^+} \bigcup_{q \in \mathbb{Z} \cap [2,\infty)} \bigcup_{p \in \mathbb{Z}} \left(\left(\frac{p}{q} - \frac{1}{q^n}, \frac{p}{q} + \frac{1}{q^n} \right) \setminus \left\{ \frac{p}{q} \right\} \right).$$

Each U_n is an open set; as its closure contains all rationals (the *p/q* from each punctured interval), it is also a dense subset of real line. Since it is the intersection of countably many such open dense sets, *L* is comeagre, that is to say, it is a *dense* G_δ set.

Irrationality Measure

The Liouville-Roth irrationality measure (irrationality exponent, approximation exponent, or Liouville–Roth constant) of a real number *x* is a measure of how "closely" it can be approximated by rationals. Generalizing the definition of Liouville numbers, instead of allowing any *n* in the power of *q*, we find the largest possible value for μ such that $0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^\mu}$ is satisfied by an infinite number of integer pairs (*p*, *q*) with $q > 0$. This maximum value of μ is defined to be the irrationality measure of *x*. For any value μ less than this upper bound, the infinite set of all rationals *p*/*q* satisfying the above inequality yield an approximation of *x*. Conversely, if μ is greater than the upper bound, then there are at most finitely many (*p*, *q*) with $q > 0$ that satisfy the inequality; thus, the opposite inequality holds for all larger values of *q*. In other words, given the irrationality measure μ of a real number *x*, whenever a rational approximation $x \cong p/q$, $p,q \in \mathbf{N}$ yields $n + 1$ exact decimal digits, we have $\frac{1}{10^n} \geq \left| x - \frac{p}{q} \right| \geq \frac{1}{q^{\mu+\varepsilon}}$

for any $\varepsilon > 0$, except for at most a finite number of "lucky" pairs (*p*, *q*).

For a rational number α the irrationality measure is $\mu(\alpha) = 1$. The Thue–Siegel–Roth theorem states that if α is an algebraic number, real but not rational, then $\mu(\alpha) = 2$. Almost allnumbers have an irrationality measure equal to 2. Transcendental numbers have irrationality measure 2 or greater. For example, the transcendental number e has $\mu(e) = 2$. The irrationality measures of π, log 2, and log 3 are at most 7.60630853, 3.57455391, and 5.125, respectively.

It has been proven that if the series $\sum_{n=1}^{\infty} \frac{\csc^2 n}{n^3}$ (where n is in radians) converges, then π's irrationality measure is most 2.5. The Liouville numbers are precisely those numbers having infinite irrationality measure.

Irrationality base

The *irrationality base* is a weaker measure rationality introduced by J.Sondow and is regarded as an irrationality measure for Liouville numbers. It is defined as follows:

Let $\propto$ be an irrational number. If there exists a real number $\beta \geq 1$ with the property that for any $\epsilon > 0$, there is a positive integer q(ϵ) such that

$$\left|\alpha - \frac{p}{q}\right| > \frac{1}{(\beta+\varepsilon)^q} \text{ for all integers } p, q \text{ with } q \geq q(\varepsilon),$$

then β is called the irrationality base of and is represented as $\beta(\propto)$

If no such β exists, then $\propto$ is called a *super Liouville number*.

Example: The series $\varepsilon_{2e} = 1 + \frac{1}{2^1} + \frac{1}{4^{2^1}} + \frac{1}{8^{4^{2^1}}} + \frac{1}{16^{8^{4^{2^1}}}} + \frac{1}{32^{16^{8^{4^{2^1}}}}} + \ldots$ is a *super Liouville number*, while the series $\tau_2 = \sum_{n=1}^{\infty} \frac{1}{{}^n 2} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^{2^2}} + \frac{1}{2^{2^{2^2}}} + \frac{1}{2^{2^{2^{2^2}}}} + \ldots$ is a Liouville number with irrationality base 2. (${}^b a$ represents tetration.)

Liouville Numbers and Transcendence

Establishing that a given number is a Liouville number provides a useful tool for proving a given number is transcendental. However, not every transcendental number is a Liouvlle number. The terms in the continued fraction expansion of every Liouville number are unbounded; using a counting argument, one can then show that there mustbeuncountablymany transcendental numbers which are not Liouville. Using the explicit continued fraction expansion of e, one can show that e is an example of a transcendental number that is not Liouville. Mahler proved in 1953 that π is another such example.

The proof proceeds by first establishing a property of irrational algebraic numbers. This property essentially says that irrational algebraic numbers cannot be well approximated by rational numbers, where the condition for "well approximated" becomes more stringent for larger denominators. A Liouville number is irrational but does not have this property, so it can't be algebraic and must be transcendental. The following lemma is usually known as **Liouville's theorem (on diophantine approximation)**, there being several results known as Liouville's theorem.Below, we will show that **no Liouville number can bealgebraic.**

Lemma: If α isanirrational number which is the root of a polynomial f of degree $n > 0$ with integer coefficients, then there exists a real number $A > 0$ such that, for all integers p, q, with $q > 0$,

$$\left|\alpha - \frac{p}{q}\right| > \frac{A}{q^n}$$

Proof of Lemma: Let M be the maximum value of $|f'(x)|$ (the absolute value of the derivative of f) over the interval $[\alpha - 1, \alpha + 1]$. Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be the distinct roots of f which differ from α. Select some value $A > 0$ satisfying

$$A < \min\left(1, \frac{1}{M}, |\alpha - \alpha_1|, |\alpha - \alpha_2|, \ldots, |\alpha - \alpha_m|\right)$$

Now assume that there exist some integers p, q contradicting the lemma. Then

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{A}{q^n} \leq A < \min\left(1, \frac{1}{M}, |\alpha - \alpha_1|, |\alpha - \alpha_2|, \ldots, |\alpha - \alpha_m|\right)$$

Then p/q is in the interval $[\alpha - 1, \alpha + 1]$; and p/q is not in $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$, so p/q is not a root of f; and there is no root of f between α and p/q.By the mean value theorem, there exists an x_0 between p/q and α such that

$$f(\alpha) - f(\tfrac{p}{q}) = (\alpha - \frac{p}{q}) \cdot f'(x_0)$$

Since α is a root of f but p/q is not, we see that $|f'(x_0)| > 0$ and we can rearrange:

$$\left|\alpha - \frac{p}{q}\right| = \frac{\left|f(\alpha) - f(\frac{p}{q})\right|}{|f'(x_0)|} = \left|\frac{f(\frac{p}{q})}{f'(x_0)}\right|$$

Now, f is of the form $\sum_{i=0}^{n} c_i x^i$ where each c_i is an integer; so we can express $|f(p/q)|$ as

$$\left|f\left(\frac{p}{q}\right)\right| = \left|\sum_{i=0}^{n} c_i p^i q^{-i}\right| = \frac{1}{q^n}\left|\sum_{i=0}^{n} c_i p^i q^{n-i}\right| \geq \frac{1}{q^n}$$

the last inequality holding because p/q is not a root of f and the c_i are integers.Thus we have that $|f(p/q)| \geq 1/q^n$. Since $|f'(x_0)| \leq M$ by the definition of M, and $1/M > A$ by the definition of A, we have that

$$\left|\alpha - \frac{p}{q}\right| = \left|\frac{f(\frac{p}{q})}{f'(x_0)}\right| \geq \frac{1}{Mq^n} > \frac{A}{q^n} \geq \left|\alpha - \frac{p}{q}\right|$$

which is a contradiction; therefore, no such p, q exist; proving the lemma.

Proof of assertion: As a consequence of this lemma, let x be a Liouville number; as noted in the article text, x is then irrational. If x is algebraic, then by the lemma, there exists some integer n and some positive real Asuch that for all p, q

$$\left|x - \frac{p}{q}\right| > \frac{A}{q^n}$$

Let r be a positive integer such that $1/(2^r) \leq A$. If we let $m = r + n$, and since x is a Liouville number, then there exist integers a, b where $b > 1$ such that

$$\left|x - \frac{a}{b}\right| < \frac{1}{b^m} = \frac{1}{b^{r+n}} = \frac{1}{b^r b^n} \leq \frac{1}{2^r}\frac{1}{b^n} \leq \frac{A}{b^n}$$

which contradicts the lemma. Hence, if a Liouville number exists, it cannot be algebraic, and therefore must be transcendental.

The idea of the Lebesgue integral is to first define a measure on subsets of ℝ. That is, we wish to assign a number m(S) to each subset S of ℝ, representing the total length that S takes up on the real number line. For example, the measure m(I) of any interval I ⊆ ℝ should be equal to its length ℓ(I).

Measure should also be additive, meaning that the measure of a disjoint union of two sets is the sum of the measures of the sets:

$$m(S \uplus T) = m(S) + m(T).$$

Indeed, if we want m to be compatible with taking limits, it should be countably additive, meaning that

$$m\left(\biguplus_{n\in\mathbb{N}} S_n\right) = \sum_{n\in\mathbb{N}} m(S_n)$$

for any sequence {Sn} of pairwise disjoint subsets of ℝ.

Of course, the measure m(ℝ) of the entire real line should be infinite, as should the measure of any open or closed ray. Thus the measure should be a function

$$m\colon \mathcal{P}(\mathbb{R}) \to [0,\infty]$$

where P(ℝ) is the power set of ℝ.

Question: Measuring Subsets of ℝ

Does there exist a function m: P(ℝ) → [0,∞] having the following properties?

1. m(I) = ℓ(I) for every interval I ⊆ ℝ.

2. For every sequence S1, S2, . . . of pairwise disjoint subsets of ℝ,

$$m\left(\biguplus_{n\in\mathbb{N}} S_n\right) = \sum_{n\in\mathbb{N}} m(S_n).$$

Surprisingly, the answer to this question is no, although it will be a while before we prove this. But it turns out that it is impossible to define a function m: $\mathcal{P}$(ℝ) → [0,∞] satisfying both of the conditions above.

The reason is that there exist certain subsets of ℝ that really cannot be assigned a measure. In fact, there is a rigorous sense in which most subsets of $\mathbb{R}$ cannot be assigned a measure. Interestingly, actual examples of this phenomenon are difficult to construct, with all such constructions requiring the axiom of choice. As a result, such poorly behaved sets are quite rare in practice, and it is possible to define a measure that works well for almost any set that one is likely to encounter.

Thus our plan is to restrict ourselves to a certain collection $\mathcal{M}$ of subsets of $\mathbb{R}$, which we will refer to as the Lebesgue measurable sets. We will then define a function m: $\mathcal{M} \to [0,\infty]$ called the Lebesgue measure, which has all of the desired properties, and can be used to define the Lebesgue integral. The following theorem summarizes what we are planning to prove.

6.5 MAIN THEOREM EXISTENCE OF LEBESGUE MEASURE

There exists a collection $\mathcal{M}$ of subsets of $\mathbb{R}$ (the measurable sets) and a function m: $\mathcal{M} \to [0,\infty]$ satisfying the following conditions:

1. Every interval $I \subseteq \mathbb{R}$ is measurable, with $m(I) = \ell(I)$.

2. If $E \subseteq R$ is a measurable set, then the complement $E^c = \mathbb{R} - E$ is also measurable.

3. For each sequence $\{E_n\}$ of measurable sets in $\mathbb{R}$, the union $\bigcup_{n\in\mathbb{N}} En$ is also measurable. Moreover, if the sets $\{E_n\}$ are pairwise disjoint,

then $m\left(\biguplus_{n\in\mathbb{N}} E_n\right) = \sum_{n\in\mathbb{N}} m(E_n).$

6.5.1Lebesgue Outer Measure

We begin by defining the Lebesgue outer measure, which assigns to each subset S of $\mathbb{R}$ an "outer measure" m^* (S). Thus m^* will be a function $m^* : \mathcal{P}(\mathbb{R}) \to [0,\infty]$

where $\mathcal{P}(\mathbb{R})$ denotes the power set of $\mathbb{R}$.

Of course, $m*$ will not be countably additive. Instead, it will have the weaker property of countable subadditivity, meaning that

$$m^*\left(\bigcup_{n\in\mathbb{N}} S_n\right) \leq \sum_{n\in\mathbb{N}} m^*(S_n)$$

for any sequence $\{S_n\}$ of subsets of ℝ. The outer measure m^* should be thought of as our first draft of Lebesgue measure. Indeed, once we determine which subsets of R are measurable, we will simply restrict m^* to the collection $\mathcal{M}$ of measurable sets to obtain the Lebesgue measure m. Thus, even though m^* is not countably additive in general, it will turn out be countably additive on the collection of measurable sets. For the following definition, we say that a collection C of subsets of ℝ covers a set S ⊆ ℝ if S ⊆ S ⋃C.

Definition: Lebesgue Outer Measure

If S ⊆ ℝ, the **(Lebesgue) outer measure** of S is defined by

$$m^*(S) = \inf\left\{\sum_{I\in\mathcal{C}} \ell(I) \;\middle|\; \mathcal{C} \text{ is a collection of open intervals that covers } S\right\}.$$

It should make intuitive geometric sense that $m^*(J) = \ell(J)$ for any interval J, though we will put off the proof of this for a little while. The difficult part is to show that if we cover an interval J with open intervals, then the sum of the lengths of the open intervals is greater than or equal to the length of J.
Note that $m^*(S)$ may be infinite if $\sum_{I\in C} \ell(I)$ is infinite for every collection C of open intervals that covers S. For example, it is not hard to see that $m^*(\mathbb{R})$ must be infinite.

Proposition 1 Properties of $m*$

Lebesgue outer measure $m*$ has the following properties:

1. $m*(\emptyset) = 0$.

2. If S ⊆ T ⊆ ℝ, then $m*(S) \leq m*(T)$.

3. If $\{S_n\}$ is a sequence of subsets of ℝ, then

$$m^*\left(\bigcup_{n\in\mathbb{N}} S_n\right) \leq \sum_{n\in\mathbb{N}} m^*(S_n)$$

PROOF Statement (1) is obvious from the definition. For (2), let S ⊆ T ⊆ ℝ, and let C be any collection of open intervals that covers T. Then C also covers S, so

$$m^*(S) \leq \sum_{I\in\mathcal{C}} \ell(I).$$

This holds for every cover C of T by open intervals, and therefore m∗ (S) ≤ m∗ (T).

For (3), let $\{S_n\}$ be a sequence of subsets of ℝ, and let $S = \bigcup_{n\in N}$ Sn. If m∗ (S_n) is infinite for some n, then by statement (2) it follows that m∗ (S) = ∞, and we are done. Suppose then that m∗ $(S_n) < \infty$ for all n. For each n, let Cn be a cover of S_n by open intervals so that

$$\sum_{I\in\mathcal{C}_n} \ell(I) \leq m^*(S_n) + \frac{\epsilon}{2^n}.$$

Then $C = \bigcup_{n\in N} C_n$ is a cover of S by open intervals, so

$$m^*(S) \leq \sum_{I\in\mathcal{C}} \ell(I) \leq \sum_{n\in\mathbb{N}} \sum_{I\in\mathcal{C}_n} \ell(I) \leq \sum_{n\in\mathbb{N}} \left(m^*(S_n) + \frac{\epsilon}{2^n}\right) = \epsilon + \sum_{n\in\mathbb{N}} m^*(S_n).$$

Since ϵ was arbitrary, statement (3) follows.

Lebesgue Measure

We are now ready to define the measurable subsets of R. There are many possible equivalent definitions of measurable sets, and the following definition is known as Carétheodory's criterion. It is not very intuitive, and we shall see equivalent definitions of measurability later on that make much more sense. The advantage of Carétheodory's criterion is that it is relatively easy to use from a theoretical perspective, and also it can be generalized to many other settings.

Definition: Lebesgue Measure A subset E of ℝ is said to be (Lebesgue) measurable if

$$m^*(T \cap E) + m^*(T \cap E^c) = m^*(T).$$

for every subset T of ℝ. In this case, the outer measure $m*(E)$ of E is called the (Lebesgue) measure of E, and is denoted m(E). The arbitrary subset T of ℝ that appears in the criterion is known as a test set. Note that

$$m*(T \cap E) + m*(T \cap E^c) \geq m*(T)$$

automatically since m∗ is subadditive

. Thus a set E is Lebesgue measurable if and only if

$$m*(T \cap E) + m*(T \cap E^c) \leq m*(T)$$

for every test set T. Note also that Carétheodory's criterion is symmetric between E and E^c . Thus a set E is measurable if and only if its complement E^c is measurable.

Proposition 2 Union of Two Measurable Sets

If E and F are measurable subsets of ℝ, then $E \cup F$ is also measurable.

PROOF :

Let $T \subseteq \mathbb{R}$ be a test set. Since E is measurable, we know that

$$m*(T) = m*(T \cap E) + m*(T \cap E^c). \qquad (1)$$

Also, if we use $T \cap (E \cup F)$ as a test set, we find tha

$$t\ m*\ T \cap (E \cup F) = m*(T \cap E) + m*\ T \cap E^c \cap F \qquad . (2)$$

Finally, since F is measurable, we know that

$$m*(T \cap E^c) = m*(T \cap E^c \cap F) + m*(T \cap E^c \cap F^c). \qquad (3)$$

Combining equations (1), (2), and (3) together yields

$$m*(T) = m*\ T \cap (E \cup F) + m*(T \cap E^c \cap F^c).$$

Since $E^c \cap F^c = (E \cup F)^c$, this proves that $E \cup F$ is measurable.

Corollary 3 Intersection of Two Measurable Sets

If E and F are measurable subsets of R, then E ∩ F is also measurable.

PROOF Since E and F are measurable, their complements E^c and F^c is also measurable. It follows that the union $E^c \cup F^c$ is measurable, and the complement of this is E ∩ F

Proposition 4 Countable Additivity Let {Ek} be a sequence of pairwise disjoint measurable subsets of R. Then the union $\biguplus$ k∈N Ek is measurable, and

$$m\left(\biguplus_{k\in\mathbb{N}} E_k\right) = \sum_{k\in\mathbb{N}} m(E_k).$$

PROOF Let T ⊆ ℝ be a test set, and let U = $\biguplus$k∈N Ek. We wish to show that m∗ (T) ≥ m∗ (T ∩ U) + m∗ (T ∩ U^c).

For each n ∈ N, let Un = $\biguplus_{k=1}^{n}$Ek. By the Proposition 2, each Un is measurable, so m∗ (T) = m∗ (T ∩ Un) + m∗ (T ∩ U_n^c).

But each Un ⊆ U, so T ∩ U_n^c ⊇ T ∩ U c ,and hence

m∗ (T) ≥ m∗ (T ∩ Un) + m∗ (T ∩ U^c).

Thus it suffices to show that m∗ (T ∩ Un) → m∗ (T ∩ U) as n → ∞.

To prove this claim, observe first that

m∗ (T ∩ Uk) = m∗ (T ∩ Uk ∩ Ek) + m∗ (T ∩ Uk ∩ E c k) = m∗ (T ∩ Ek) + m∗ (T ∩ Uk−1).

for each k. By induction, it follows that

$$m^*(T \cap U_n) = \sum_{k=1}^{n} m^*(T \cap E_k)$$

for each n. Then

$$\sum_{k=1}^{n} m^*(T \cap E_k) = m^*(T \cap U_n) \leq m^*(T \cap U) \leq \sum_{k\in\mathbb{N}} m^*(T \cap E_k),$$

where the last inequality follows from the countable subadditivity of m∗ .

By the squeeze theorem, we conclude that

$$\lim_{n\to\infty} m^*(T \cap U_n) = m^*(T \cap U) = \sum_{k\in\mathbb{N}} m^*(T \cap E_k),$$

which proves that U is measurable. Moreover, in the case where T = ℝ, the last equation gives

$$m(U) = \sum_{k\in\mathbb{N}} m(E_k).$$

Corollary 5 Countable Union of Measurable Sets If $\{E_k\}$ is any sequence of measurable subsets of R, then the union ⋃k∈N E_k is measurable.

PROOF Let Un = $\bigcup_{k=1}^{n}$ Ek for each k, and let Fn = Un − Un−1, with F1 = U1.

By Proposition 2, we know that each Un is measurable, and thus

$F_n = U_n \cap U_{n-1}^c$ is measurable by Corollary 3. But the sets {Fn} are disjoint, and

$$\biguplus_{n\in\mathbb{N}} F_n = \bigcup_{k\in\mathbb{N}} E_k$$

so $\bigcup_{k\in\mathbb{N}} E_k$ is measurable.

6.5.2The Geometry Of Intervals

All that remains in proving the desired properties of Lebesgue measure is to show that intervals in ℝ are measurable, with m(I) = ℓ(I) for any interval I. Unlike all of the work so far, proving this requires exploiting the geometry of intervals in a significant way. We begin with the following proposition.

Proposition 6 Intervals are Measurable

Every interval J in ℝ is Lebesgue measurable.

PROOF Since each interval in ℝ is the intersection of two rays, it suffices to prove that each ray in ℝ is measurable.

Let ℝ be a ray in ℝ, and let $T \subseteq ℝ$ be a test set. We wish to prove that

$$m*(T) \geq m*(T \cap R) + m*(T \cap R^c)$$

If $m*(T) = \infty$ then we are done, so suppose that $m*(T) < \infty$. Let > 0, and let C be a cover of T by open intervals so that

$$\sum_{I \in \mathcal{C}} \ell(I) \leq m^*(T) + \frac{\epsilon}{2}.$$

Since the sum $\sum_{I \in C} \ell(I)$ is finite, C must be countable (see the appendix on sums).

Let $\{I_1, I_2, \ldots\}$ be an enumeration of the elements of C, where we set $I_n = \emptyset$ for $n > |C|$ if C is finite. Then each of the intersections $I_n \cap R$ and $In \cap R^c$ is an interval, with $\ell(I_n \cap R) + \ell(I_n \cap R^c) = \ell(I_n)$. For each n, let Jn and Kn be open intervals containing $I_n \cap R$ and $I_n \cap R^c$, respectively, such that

$\ell(J_n) \leq \ell(I_n \cap R) + 2\ n{+}2$ and $\ell(K_n) \leq \ell(I_n \cap R^c) + 2\ n{+}2$.

Then $\{J_n\}n \in N$ is a cover of $T \cap R$ by open intervals, and $\{K_n\}n \in N$ is a cover of $T \cap R^c$ by open intervals,

so

$$\begin{aligned} m^*(T \cap R) + m^*(T \cap R^c) &\leq \sum_{n \in \mathbb{N}} \ell(J_n) + \sum_{n \in \mathbb{N}} \ell(K_n) \\ &\leq \sum_{n \in \mathbb{N}} \left(\ell(I_n \cap R) + \frac{\epsilon}{2^{n+2}} \right) + \sum_{n \in \mathbb{N}} \left(\ell(I_n \cap R^c) + \frac{\epsilon}{2^{n+2}} \right) \\ &= \frac{\epsilon}{2} + \sum_{n \in \mathbb{N}} \ell(I_n) \leq m^*(T) + \epsilon. \end{aligned}$$

Since ϵ was arbitrary, this proves the desired inequality. All that remains is to prove that the measure of any interval is equal to its length. For this we need the famous Heine-Borel theorem, which we will state and prove next. Those familiar with point-set topology should recognize this theorem as a special case of the statement that closed intervals in ℝ are compact. In fact, the notion of compactness in point-set topology arose as a generalization of this theorem.

6.5.3Heine-Borel Theorem

Let [a, b] be a closed interval in R, and let C be a family of open intervals that covers [a, b]. Then there exists a finite subcollection of C that covers [a, b].

Proof:

Let S be the set of all points s ∈ [a, b] for which the interval [a, s] can be covered by some finite subcollection of C. Note that a ∈ S, since the interval [a, a] is just a single point. Our goal is to prove that b ∈ S. Let x = sup(S). Since S ⊆ [a, b], we know that x ∈ [a, b]. Therefore, there exists an interval (c, d) ∈ C that contains x. Since c < x, there is some point s ∈ S that lies between c and x. Let $\{(c_1, d_1), \ldots,(c_n, d_n)\}$ be a finite subcollection of C that covers [a, x]. Then the collection $\{(c_1, d_1), \ldots,(c_n, d_n),(c, d)\}$ covers [a, x], which proves that x ∈ S.

Moreover, if x < b, then there exists an > 0 such that

x + ∈ [a, b] and x + ∈ (c, d).

Then the collection $\{(c_1, d_1), \ldots,(c_n, d_n),(c, d)\}$ covers [a, x +], which proves that x + ∈ S, a contradiction since x is the supremum of S. We conclude that x = b, and therefore b ∈ S. In addition to the Heine-Borel theorem, the following proof will use the Riemann integral and characteristic functions. If S is any subset of ℝ, the characteristic function (or indicator function) for S is the function $\chi S : \mathbb{R} \to \mathbb{R}$ defined by

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{if } x \notin S. \end{cases}$$

Note that if *I* is an interval then

Proposition 8 Measure of an Interval

If *J* is any interval in ℝ, then m(J) = ℓ(J).

PROOF Note first that, for every $\epsilon > 0$, there exists an open interval J′ containing J so that $\ell(J') \leq \ell(J) + .\ \ell(J') \leq \ell(J) +$. Since ϵ was arbitrary, it follows that $m(J) \leq \ell(J)$. Now let C be any collection of open intervals that covers J. Let > 0, and let K be a closed subinterval of J such that $\ell(K) \geq \ell(J) - \epsilon$. By the Heine-Borel theorem, there exists a finite subcollection $\{I_1, . . . , I_n\}$ of C that covers K. Then

$$\chi I_1 + \cdot \cdot \cdot + \chi I_n \geq \chi K$$

so

$$\begin{aligned}\sum_{I \in \mathcal{C}} \ell(I) &\geq \ell(I_1) + \cdots + \ell(I_n) = \int_{-\infty}^{\infty} \chi_{I_1}(x)\, dx + \cdots + \int_{-\infty}^{\infty} \chi_{I_n}(x)\, dx \\ &= \int_{-\infty}^{\infty} (\chi_{I_1}(x) + \cdots + \chi_{I_n}(x))\, dx \geq \int_{-\infty}^{\infty} \chi_K(x)\, dx = \ell(K) \geq \ell(J) - \epsilon.\end{aligned}$$

Since ϵ was arbitrary, it follows that

$$\sum_{I \in \mathcal{C}} \ell(I) \geq \ell(J)$$

which proves that $m(J) \geq \ell(J)$.

6.6 LETS SUM UP

In this unit, we have covered the following points.

1. We have defined a-algebra of subsets of a set X.

2. We have defined out measure and discussed how to compute the outer

measure of set.

3. We have discussed how to check whether a set is measurable or not.

4. We have given an example of a set which is not measurable.

5. We have defined measurable functions.

6.7 KEYWORD

HEINE-BORAL THEOREM

LEBESQUE

SUBCOLLECTION

ENUMARATION

SUBCOVER

COUNTABLE

6.8 QUESTION FOR REVIEW

1. If $\{E_n\}$ is a sequence of measurable sets, prove that the intersection T $_{n\in\mathbb{N}}$ E_n is measurable.

2. Prove that if $S \subseteq R$ and $m*(S) = 0$, then S is measurable.

3. a) If $E \subseteq F$ are measurable sets, prove that $F - E$ is measurable. b) Prove that if $m(E) < \infty$ then $m(F - E) = m(F) - m(E)$.

4. If E and F are measurable sets with finite measure, prove that $m(E \cup F) = m(E) + m(F) - m(E \cap F)$.

5. Suppose that $E \subseteq S \subseteq F$, where E and F are measurable. Prove that if $m(E) = m(F)$ and this measure is finite, then S is measurable as well.

6. Prove that every countable subset of $\mathbb{R}$ is measurable and has measure zero.

7. Given a nested sequence $E1 \subseteq E2 \subseteq \cdot \cdot \cdot$ of measurable sets, prove that

$$m\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \sup_{n\in\mathbb{N}} m(E_n).$$

6.9 SUGGESTED READINGS & REFERENCE

1. Oxtoby, John C. (1980). Measure and Category. Graduate Texts in Mathematics. **2** (Second ed.). New York-Berlin: Springer-Verlag. doi:10.1007/978-1-4684-9339-9. ISBN 0-387-90508-1. MR 0584443.
2. ^ Olsen, Lars Ole Rønnow; Renfro, Dave L. (2006). "On the exact Hausdorff dimension of the set of Liouville numbers. II". Manuscripta Mathematica. **119** (2): 217–224. doi:10.1007/s00229-005-0604-z. MR 2215968.
3. ^ Jump up to:[a b c d e f] Bugeaud, Yann (2012). Distribution modulo one and Diophantine approximation. Cambridge Tracts in Mathematics. **193**. Cambridge: Cambridge University Press. doi:10.1017/CBO9781139017732. ISBN 978-0-521-11169-0. MR 2953186. Zbl 1260.11001.
4. ^ Zudilin, Wadim (2004). "An essay on the irrationality measure of π and other logarithms". Chebyshevskii Sbornik (in Russian). **5** (2(10)): 49–65. arXiv:math/0404523. Bibcode:2004math......4523Z. MR 2140069. Zbl 1140.11036.
5. ^ Max A. Alekseyev, On convergence of the Flint Hills series, arXiv:1104.5100, 2011.
6. ^ Weisstein, Eric W. "Flint Hills Series". MathWorld.
7. ^ Sondow, Jonathan. (2004). Irrationality Measures, Irrationality Bases, and a Theorem of Jarnik. https://arxiv.org/abs/math/0406300
8. ^ The irrationality measure of π does not exceed 7.6304, according to Weisstein, Eric W. "Irrationality Measure". MathWorld.
9. Lebesgue, Henri (1904). Leçons sur l'Intégration et la recherche des fonctions primitives. Paris: Gauthier-Villars.

10. Lebesgue, Henri (1910). "Sur l'intégration des fonctions discontinues". Annales Scientifiques de l'École Normale Supérieure. **27**: 361–450.
11. Wheeden, Richard L.; Zygmund, Antoni (1977). Measure and Integral – An introduction to Real Analysis. Marcel Dekker.
12. Oxtoby, John C. (1980). Measure and Category. Springer Verlag.
13. Stein, Elias M.; Shakarchi, Rami (2005). Real analysis. Princeton Lectures in Analysis, III. Princeton, NJ: Princeton University Press. pp. xx+402. ISBN 0-691-11386-6. MR2129625
14. Benedetto, John J.; Czaja, Wojciech (2009). Integration And Modern Analysis. Birkhäuser Advanced Texts. Springer. pp. 361–364. ISBN 0817643060.

6.10 ANSWERS TO CHECK YOUR PROGRESS

1. Check Section 6.1

2 Check section 6.3

UNIT -7: LEBESGUE OUTER MEASURE

STRUCTURE

7.1 INTRODUCTION

We are now ready to define the Lebesgue outer measure set function. This is a set function defined for all subsets of ℝ.

The Lebesgue outer measure (or outer measure) of a set A ⊆ Ris given by $m^*(A) = glb \sum_{n=1}^{\infty} \ell(I_n)$, where the infimum is taken over all the possible countable collection of open intervals I_n such that

$$A \subseteq \bigcup_{n=1}^{\infty} I_n.$$

Definition: The **Lebesgue Outer Measure Function** is the function $m^* : \mathcal{P}(\mathbb{R}) \to [0, \infty) \cup \{\infty\}$ defined for all sets $E \in \mathcal{P}(\mathbb{R})$ by

$$m^*(E) = \inf\left\{ \sum_{n=1}^{\infty} l(I_n) : E \subseteq \bigcup_{n=1}^{\infty} I_n \text{ and } \{I_n = (a_n, b_n)\}_{n=1}^{\infty} \right\}.$$

The **Lebesgue Outer Measure of the Set** E is $m*(E)$.

In other words, if $E \subseteq P(\mathbb{R})$ *then* $m*(E)$ *is defined by taking all open-interval covers of E, summing the lengths of all of the open intervals in those covers, and then taking the infimum of these values.*

Sometimes for brevity we simply say "outer measure" to refer to the Lebesgue outer measure of a set.

We now describe some important properties of the Lebesgue outer measure function.

- Measurement has been a special interest for mathematicians and scientists since the early days of civilization.
- Two thousand years ago, the classical geometers of Greece made profound contributions to the study of measurement.
- To determine the area of a circle, the Greeks constructed sequences of inscribed and circumscribed regular polygons, with the number of sides tending to infinity.
- This gives a sequence of lower and upper estimates of the area of the circle, and the area of the circle is defined to be the common limit as the number of sides tends to infinity.
- This procedure, known as the Method of Exhaustion, was formulated by Euxodus of Cnidos (408-355 B.C.E.)., and was developed systematically by Archimedes (287-212 B.C.E.). Outline Introduction Objective Lebesgue Outer Measure Fundamental Property References
- Although the Riemann integral suffices in most daily situations, it fails to meet our needs in several important ways. First, the class of Riemann integrable functions is relatively small. Second and related to the first, the Riemann integral does not have satisfactory limit properties. Third, all Lp spaces except L∞ fails to complete under Riemann integration.
- Later we will see how can we overcome these limitation using more abstract spaces.
- The purpose of this and subsequent lessons is to provide a concise introduction to Measure theory, in the context of abstract measure spaces.

7.1.1 LEBESGUE OUTER MEASURE THEOREM

Theorem 1: Let $E \in P(R)$ be a finite set. Then $m_*(E)=0$.

Proof: Let $E=\{x_1,x_2,...,x_n\}$. Then for every $\epsilon>0$ and for all $k\in\{1,2,...,n\}$ we have that xk $x_k \in \left(x_k - \frac{\epsilon}{2}, x_k + \frac{\epsilon}{2}\right)$. So:

$$E \subseteq \bigcup_{k=1}^{n}\left(x_k - \frac{\epsilon}{2}, x_k + \frac{\epsilon}{2}\right)$$

Furthermore, the length corresponding to the open interval cover of E for each given ϵ is given by:

$$\sum_{k=1}^{n} l\left(\left(x_k - \frac{\epsilon}{2}, x_k + \frac{\epsilon}{2}\right)\right) = \sum_{k=1}^{n} \epsilon = n\epsilon$$

So for all $\epsilon>0$ we have that

$$m^*(E) < n\epsilon$$

Since n is fixed, the inequality above implies that $m*(E)=0$. ■

7.1.2 Theorem 2 (Monotonicity Of The Lebesgue Outer Measure)

Let $A,B\in P(\mathbb{R})$. If $A\subseteq B$ then $m*(A)\leq m*(B)$.

Proof: Let $A,B\in P(\mathbb{R})$ be such that $A\subseteq B$.

$\{I_n = (a_n, b_n)\}_{n=1}^{\infty}$ is such that $B \subseteq \bigcup_{n=1}^{\infty} I_n$ then $A \subseteq \bigcup_{n=1}^{\infty} I_n$. Therefore:

$$\left\{\sum_{n=1}^{\infty} l(I_n) : A \subseteq \bigcup_{n=1}^{\infty} I_n \text{ and } \{I_n = (a_n, b_n)\}_{n=1}^{\infty}\right\} \supseteq \left\{\sum_{n=1}^{\infty} l(I_n) : B \subseteq \bigcup_{n=1}^{\infty} I_n \text{ and } \{I_n = (a_n, b_n)\}_{n=1}^{\infty}\right\}$$

Example Let A be any countable set. Therefore, it can be expreseed as

$$A = \{x_1, x_2, x_3, \ldots\ldots, x_n, \ldots\ldots\} = \bigcup_{n=1}^{\infty} \{x_n\}.$$

Then,

$$\begin{aligned} m^*(A) = m^*\left(\bigcup_{n=1}^{\infty}\{x_n\}\right) &\leq \sum_{n=1}^{\infty} m^*\{x_n\} \\ &= m^*(x_1) + .. + m^*(x_n) + .. \\ &= 0. \end{aligned}$$

Example : Suppose, A is countable. But, then it can be written as a sequence

$$A = \{x_1, x_2, x_3, \ldots\ldots, x_n, \ldots\ldots\} = \bigcup_{n=1}^{\infty} \{x_n\}.$$

And in that case, m∗ (A) = 0. A contradiction to the given fact that m ∗ (A) ≠ 0. Hence A should be uncountable.

Note. We now introduce a type of measure with an eye towards the requirement that m(I) =ℓ(I) for intervals (at least, for open intervals).

Definition. Let A⊂ℝ and let $\{I_n\}$ represent a countable collection of bounded open intervals such that A ⊂ $\bigcup I_n$. The outer measure of A is

$$m^*(A) = \inf_{A \subset \cup I_n} \left\{ \sum_{n=1}^{\infty} \ell(I_n) \right\}$$

where the infimum is taken over all such open interval coverings of A.

Note. Since $m^*(A)$ is defined as an infimum, then $m^*(A)$ is defined for every A ϵ P (ℝ). $m^*(\emptyset) = 0$ and if A is finite in cardinality then $m^*(A) = 0$. Also, if A ⊂ B then $m^*(A) \le m^*(B)$ (i.e., m^* satisfies monotonicity).

7.1.3 SOME EXAMPLE ON LEBESGUE OUTER MEASURE

So far we have shown the following properties regarding the Lebesgue outer measure:

1. For any finite or countably infinite subset E **of** R**:**

$$m^*(E)=0$$

m^* **has the monotonicity property. If** $A \subseteq B$ **then:**

$$m(A) \le m(B)$$

If I is any interval then:

$$m^*(I) = l(I)$$

m^* is translation invariant. For any subset E of R and for any $a \in$ R we have that:

$$m^*(E+a) = m^*(E)$$

m^* is countably subadditive. For any sequence $(A_n)_{k=1}^{\infty}$ of subsets of $\mathbb{R}$ we have that:

$$m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} m^*(A_n)$$

We now look at some example problems involving the Lebesgue outer measure.

Example 1

Prove that $[0,1]$ is an uncountable subset of $\mathbb{R}$.

If $[0,1]$ were countable then $m^*([0,1])=0$. But $[0,1]$ is an interval and so $m^*([0,1])=l([0,1])=1$. Therefore $[0,1]$ must be an uncountable subset of $\mathbb{R}$.

Example 2

Let I be the set of all irrational numbers contained in the interval [0,1]. Prove that $m^*(I)=1$.

The set of all rational numbers contained in $[0,1]$ is a countable set and so by countable subadditivity we have that:

$$1 = m^*([0,1]) = m^*(I \cup ([0,1] \setminus I)) \le m^*(I) + m([0,1] \setminus I) = m^*(I)$$

But since $I \subseteq [0,1]$ we have the monotonicity of the Lebesgue outer measure that:

$$m^*(I) \le m^*([0,1]) = 1$$

Therefore we conclude that $m^*(I)=1$.

Example 3

Let $A,B \subseteq \mathbb{R}$. Prove that if $m^*(A)=0$ then **$m^*(A \cup B) = m^*(B)$**.

Since $B \subseteq A \cup B$ we have by the monotonicity property of the Lebesgue outer measure that:

$$m^*(B) \leq m^*(A \cup B)$$

Furthermore by the countable subadditivity of the Lebesgue outer measure we have that:

$$m^*(A \cup B) \leq m^*(A) + m^*(B) = 0 + m^*(B) = m^*(B)$$

Therefore we conclude that $m^*(A \cup B) = m^*(B)$.

Definition: For any open interval I = (a, b), define λ(I) = b − a.

Recall. A set of real numbers G is open if and only if it is a countable disjoint union of open intervals:

$$G = \cup_{k=1}^{\infty} I_k \text{ where } I_j \cap I_k = \emptyset \text{ if } j \neq k$$

where each I_k is an open interval.

Definition: For the above open set of real numbers $G = \cup_{k=1}^{\infty} I_k$ define

$$\lambda(G) = \sum_{k=1}^{\infty} \lambda(I_k).$$

If one of the I_k is unbounded, define λ(G) = ∞ and if G = ∅ define λ(G) = 0.

Definition: Let E be a bounded closed set with a = glb(E) and b = lub(E) (that is, [a, b] is the smallest closed interval containing E). Define

$$\lambda(E) = b - a - \lambda((a, b) \setminus E).$$

Notice: If E is closed, then (a, b) \ E = (a, b) ∩ E^c is open. Also, we get by rearranging:

$$\lambda(E) + \lambda((a, b) \setminus E) = b - a.$$

Note: We have λ defined on any open set or any closed and bounded set. We now use λ defined on the open sets to define outer measure, identical to Royden and Fitzpatrick's approach.

Definition: Let E be an arbitrary subset of ℝ.

Let $\lambda^*(E) = \inf\{\lambda(g) \mid E \subset G, G \text{ is open}\}$.

Then $\lambda^*(E)$ is called the Lebesgue outer measure of E.

Note. By definition, for open G, $\lambda^*(G) = \lambda(G)$.

Theorem: For every $E \subset R$, there exists a G_δ set G such that $E \subset G$ and $\lambda^*(E) = \lambda^*(G)$. G is called a measurable cover for E.

Proof: This is a result in Royden and Fitzpatrick (Theorem 2.11(ii)).

Note. Since for open G (with the notation from above), $\lambda(G) = \sum_{k=1}^{\infty} \lambda(I_k)$, we immediately have

$$\lambda^*(E) = \inf\left\{\sum_{k=1}^{\infty} \lambda(I_k) \mid E \subset \bigcup_{k=1}^{\infty} I_k,\ \text{each } I_k \text{ an open interval}\right\}.$$

This is the same as Royden and Fitzpatrick's definition of outer measure μ^*. As previously mentioned, we show in Real Analysis 1 that $\lambda^* = \mu^*$ is (1) translation invariant, (2) monotone, (3) the outer measure of an interval is its length, and (4) countably subadditive. 9

Note: It would seem that λ^* should do for a measure. However, λ^* is not countably additive. In fact, there are disjoint sets E_1 and E_2 such that

$$\lambda(E_1 \cup E_2) = \lambda^*(E_1) + \lambda^*(E_2)$$

does not hold. Specific examples of such sets are seen with the construction of a nonmeasurable set (climaxing in the "offensive" Banach-Tarski Paradox).

Definition: Let E be an arbitrary subset of R. Let

$\lambda_*(E) = \sup\{\lambda(F) \mid F \subset E, F \text{ is compact}\}$.

Then $\lambda_*(E)$ is called the Lebesgue *inner measure* of E.

Note: By definition, for compact F, $\lambda_*(F) = \lambda(F)$.

Note: Similar to the proofs for μ^*, we can show that λ_* is:

(1) translation invariant ($\lambda_*(E + x) = \lambda_*(E)$ for all $x \in R$),

(2) monotone ($A \subset B$ implies $\lambda_*(A) \leq \lambda_*(B)$),

(3) the inner measure of an interval is its length: $\lambda_*(I) = \ell(I)$ for all intervals $I \subset \mathbb{R}$, and

(4) countably superadditive

$$\lambda_*\left(\bigcup_{k=1}^{\infty} E_k\right) \geq \sum_{k=1}^{\infty} \lambda_*(E_k).$$

Theorem: For every $E \subset R$, there exists an $F\sigma$ set F such that $F \subset E$ and $\lambda_*(F) = \lambda_*(E)$. F is called a measurable kernal of g.

Proof: First, suppose $\lambda_*(E) = m < \infty$. Since

$\lambda_*(E) = \sup\{\lambda(F) \mid F \subset E, F \text{ is compact}\}$,

then by definition of supremum, for all $\varepsilon k = 1/k$, $k \in N$, there is a compact set Fk such that $m \geq \lambda(Fk) > m - 1/k$. Consider the set $F = \cup\ \infty\ k=1Fk$. Since each Fk is compact (and therefore closed), then F is a countable union of closed sets—i.e., F is an $F\sigma$ set. Also, $Fk \subset F \subset E$ for all $k \in N$. Therefore, by monotonicity of λ_*

$$m - \frac{1}{k} = \lambda_*(F_k) \leq \lambda_*(F) \leq \lambda_*(E) = m$$

for all $k \in N$, and hence $\lambda_*(F) = \lambda_*(E)$.

Second, suppose $\lambda_*(E) = \infty$. Then for all $k \in N$ there is a compact set F_k such that $\lambda_*(F_k) > k$ from the supremum definition of $\lambda_*(E)$. Again, take $F = \cup F_k$ and F is an $F\sigma$ set with

$\lambda_*(F) = \lambda_*(\cup F_k) \geq \sum \lambda_*(F_k) = \infty = \lambda_*(E)$

where the inequality part follows from the countable super additivity of λ_*

Theorem: If F is a compact set, then $\lambda^* (F) = \lambda_*(F)$. In the next section, we will see that this is the definition of measurable. So every compact set F is measurable.

Proof: Let [a, b] be the smallest interval containing F. We know that $(a, b) = ((a, b) \setminus F) \cup F$ and since λ^* is countably additive,

$\lambda^* ((a, b)) = \lambda^* (((a, b) \setminus F) \cup F) = \lambda^* ((a, b) \setminus F) + \lambda * (F)$

or

$\lambda^* (F) = \lambda^* ((a, b)) - \lambda^* ((a, b)\setminus) = b - a - \lambda^* ((a, b) \setminus F) = \lambda(F) = \lambda^*(F)$.

So F is measurable.

Note: We cannot use intervals (directly) in the definition of inner measure; since set E may not have any subsets which are intervals (consider $\mathbb{Q}$ or $\mathbb{R} \setminus \mathbb{Q}$). However, every set has a compact subset (since, trivially, the empty set is compact and has outer measure 0).

Theorem: Let [a, b] be the smallest interval containing set E. Then

$\lambda_*(E) = b - a - \lambda^* ([a, b] \setminus E)$.

Proof: First, let $F \subset E$ be compact. Then $[a, b] \setminus F$ is open and $[a, b] \setminus E \subset [a, b] \setminus F$. then

$\lambda(F) = b - a - \lambda([a, b] \setminus F)$ (definition of λ for a compact set)

$\leq b - a - \inf\{\lambda(G) \mid [a, b] \setminus E \subset G, G \text{ is open}\}$

(definition of infimum since $[a, b] \setminus F$ is one specific such open G)

$= b - a - \lambda^* ([a, b] \setminus E)$ (definition of λ^*).

Since $F \subset E$ was arbitrary, taking a suprema over all such F yields

$$\lambda_*(E) \leq b - a - \lambda^* ([a, b] \setminus E).$$

We now need to reverse this inequality.

Second, let $[a, b] \setminus E \subset G$ where G is open. Then $[a, b] \setminus G$ is compact and $[a, b] \setminus G \subset E$. Then

$$b - a - \lambda(G) \leq b - a - \inf\{\lambda(G) \mid [a, b] \setminus E \subset G, G \text{ is open}\}$$
(definition of infimum)

$$= b - a - \lambda^* ([a, b] \setminus E) \text{ (definition of } \lambda^*)$$

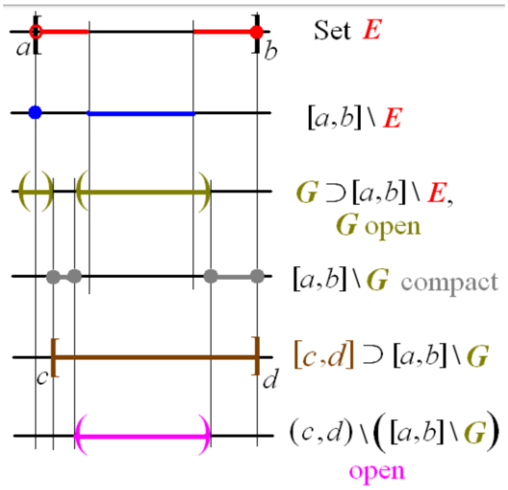
Or

$\lambda(E) \geq \lambda([a, b] \setminus G)$ (definition of supremum since $[a, b] \setminus G$ is one specific such compact set)

$= (c, d) - \lambda((c, d) \setminus ([a, b] \setminus G)$

where $[c, d]$ is the smallest closed interval containing $[a, b] \setminus G$)

$\geq (b - a) - \lambda((c, d) \setminus ([a, b] \setminus G))$ (since $[c, d] \subset [a, b]$).



If a, b ∈ E, then (c, d) = (a, b) and WLOG we have G ⊂ (a, b), so (c, d) \ ([a, b] \ G) = (a, b) \ ([a, b] \ G) = (a, b) \ ((a, b) \ G) = G.

Then

$\lambda_*(E) \geq b - a = \lambda((c, d) \setminus ([a, b] \setminus G)) = b - a - \lambda(G)$

where G is open and [a, b]\E ⊂ G. Since G was arbitrary (we have G ⊂ (a, b) WLOG), taking the infimum over all such G gives

$\lambda_* (E) \geq b - a - \lambda^* ([a, b] \setminus E)$.

Therefore when a, b ∈ E (i.e., when E contains its lub and glb),

$\lambda_* (E) = b - a - \lambda^* ([a, b] \setminus E)$.

If a is not in E, we see that [a, b] \ E differs from [a, b] \ (E ∪ {a}) by only one point. Hence, from an ε-argument, we can show that $\lambda^* ([a, b] \setminus E) = \lambda^* ([a, b] \setminus (E \cup \{a\}))$ (and similarly if neither a nor b is in E) and the result follows for arbitrary E.

Note: We will define a set to be Lebesgue measurable by always appealing to bounded portions of the set. Therefore the equation

λ_* (E) = b − a − λ^* ([a, b] \ E) has some implication even for unbounded sets. The important observation here is that even if we approach Lebesgue measure from an inner measure/outer measure perspective, we see that the inner measure is ultimately dependent only on the outer measure. Therefore, there is a degree of redundance in the introduction of inner measure at least as long as the above equation holds (and this is where the Carathéodory splitting condition arises in Royden and Fitzpatrick's development).

Check your Progress -1

Q.1 Give example of On Lebesgue Outer Measure

____

____

7.2 LEBESGUE MEASUREBILITY

Definition Let E be a bounded subset of $\mathbb{R}$, and let λ^* (E) and λ_*(E) denote the outer and inner measures of E.

$$\text{If } \lambda_* (E) = \lambda^* (E)$$

then we say that E is Lebesgue measurable with Lebesgue measure $\lambda(E) = \lambda^* (E)$. If E is unbounded, we say that E is Lebesgue measurable if $E \cap I$ is Lebesgue measurable for every finite interval I and again write $\lambda(E) = \lambda^* (E)$.

Note: Henri Lebesgue (1875–1941) was the first to crystallize the ideas of measure and the integral studied in Part 1 of our Real Analysis 1 class. In his doctoral dissertation, *Intégrale, Longueur, Aire* ("Integral, Length, Area") of 1902, he presented the definitions of inner and outer measure equivalent to the approach of Bruckner, Bruckner, and Thomson given here. His definition of "measurable" is the same as the previous definition. Lebesgue published his results in 1902, with the same title as his dissertation, in Annali di Matematica Pura ed

Applicata, Series 3, VII(4), 231–359. You can find this online (in French, or course) at https://archive.org/stream/annalidimatemat01unkngoog#page/n252/mode/2up.

Carathéodory introduced his splitting condition in 1914. His approach is to outer measure and measurability in a more abstract setting. His results appeared in Uber das lineare Mass von Punktmengen- eine Verallgemeiner ¨ ung des L¨angenbegriffs [“About the linear measure of sets of points - a generalization of the concept of length”] *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* [“News of the Society of Sciences in Göttingen , Mathematics and Physical Class”] (1914), 404–426. Carathéodory’s original paper can be found online at http://gdz.sub.uni-goettingen.de/dms/load/img/?PID=GDZPPN002504006. 15

Theorem: λ_* is monotone. That is, if $E_1 \subset E_2$ then $\lambda_*(E_1) \leq \lambda_*(E_2)$.

Proof: Let $E_a \subset E_2$. Since every compact set F which is a subset of $_q$ is also a subset of E_2, then

$$\lambda_*(E_1) = \sup\{\lambda(F) \mid F \subset E_1, F \text{ compact}\}$$

$$\leq \sup\{\lambda(F) \mid F \subset E_2, F \text{ compact}\} = \lambda^* (E_2)$$

(since the second supremum is taken over a larger collection of real numbers than the first supremum).

Theorem: If $\{E_k\}$ is a disjoint sequence of subsets of $\mathbb{R}$, then

$$\lambda_*\left(\bigcup_{k=1}^{\infty} E_k\right) \geq \sum_{k=1}^{\infty} \lambda_*(E_k).$$

This property is called countable superadditivity.

Proof: Let ε > 0. By the definition of $\lambda_*(E_k)$ in terms of a supremum, for each $k \in N$ there exists a compact set $F_k \subset E_k$ such that

$$\lambda_*(E_k) - \frac{\varepsilon}{2^k} \le \lambda_*(F_k) = \lambda(F_k),$$

a property of supremum. Next,

$$\begin{aligned}\lambda_*\left(\bigcup_{k=1}^{n} E_k\right) &\ge \lambda_*\left(\bigcup_{k=1}^{n} F_k\right) \text{ (by the monotonicity of } \lambda_*) \\ &= \lambda\left(\bigcup_{k=1}^{n} F_k\right) \text{ (since each } \cup F_k \text{ is compact and so measurable)}\end{aligned}$$

$$\begin{aligned} &= \sum_{k=1}^{n} \lambda(F_k) \text{ (since } \lambda \text{ is countably additive)} \\ &\ge \sum_{k=1}^{n} \left(\lambda_*(E_k) + \frac{\varepsilon}{2^k}\right) \\ &= \sum_{k=1}^{n} \lambda_*(E_k) + \varepsilon\left(\sum_{k=1}^{n} \frac{1}{2^k}\right)\end{aligned}$$

This holds for all n, so

$$\lambda_*\left(\bigcup_{k=1}^{\infty} E_k\right) \ge \sum_{k=1}^{\infty} \lambda_*(E_k) + \varepsilon.$$

Next, ε was arbitrary, so

$$\lambda_*\left(\bigcup_{k=1}^{\infty} E_k\right) \ge \sum_{k=1}^{\infty} \lambda_*(E_k).$$

Note.: If $E \subset \mathbb{R}$ is a bounded measurable set, and [a, b] is the smallest interval containing E, then

$\lambda_*(E) = (b - a) - \lambda^*([a, b] \setminus E)$ by Theorem 3.4

or $\lambda^*(E) = \lambda^*([a, b]) - \lambda^*([a, b] \setminus E)$

or $\lambda^*([a, b]) = \lambda^*(E) + \lambda^*([a, b] \setminus E)$. (1)

Recall the Carathéodory splitting condition from Royden and Fitzpatrick:

$$\lambda^* (X) = \lambda^* (A) + \lambda^* (X \setminus A)$$

Equation (1) is simply the splitting condition applied to the set A = [a, b]! If E is measurable and unbounded, then the condition of Lebesgue measurability implies that the splitting condition must be satisfied for all intervals. (By the additivity of λ^* , we can replace interval [a, b] with any interval and say the same thing about unbounded mearsurable sets.)

Note. Clearly, the splitting condition implies (1) and so Royden and Fitzpatrick's approach implies the inner/outer measure approach to defining Lebesgue measure. We now need to show that the inner/outer measure approach implies Royden and Fitzpatrick's approach and the Carathéodory splitting condition. This is accomplished in the following theorem.

Theorem: Let $E \subset \mathbb{R}$ is a bounded measurable set (i.e., $\lambda_* (E) = \lambda^* (E)$) and let [a, b] be the smallest interval containing E. Then for any set $A \subset \mathbb{R}$ we have

$$\lambda^* (A) = \lambda^* (A \cap E) + \lambda^* (A \setminus E).$$

Proof: Let $E \subset \mathbb{R}$ be a bounded measurable set and let [a, b] be the smallest interval containing set E. Let A be any subset of [a, b]. As already discussed in previous Theorem there is a G_δ set $G \supset A$ (called a measurable cover of A) such that $\lambda^* (G) = \lambda^* (A)$. Since $A \subset [a, b]$ and set G is G_δ, then WLOG we have $G \subset [a, b]$: Since

$[a, b] = \bigcap_{n=1}^{\infty} \left[a + \frac{1}{n}, b - \frac{1}{n}\right]$ is G_δ and, if G is not a subset of [a, b], the set $G \cap [a, b]$ is a G_δ subset of [a, b] and $A \subset G \cap [a, b]$. By monotonicity of λ^* , we have

$$\lambda^* (A) \leq \lambda^* (A \cap E) + \lambda^* (A \setminus E).$$

So we only need to show that

$$\lambda^* (A) \geq \lambda^* (A \cap E) + \lambda^* (A \setminus E).$$

Notice that

$$[a, b]\setminus G = [a, b]\cap G^c = ([a, b]\cup([a, b]\setminus E))\cap G^c = ([a, b]\cap G^c)\bigcup(([a, b]\setminus E)\cap G^c)$$

$$= ([a, b] \setminus G)\bigcup(([a, b] \setminus E) \setminus G)$$

and so by monotonicity of λ^*

$$\lambda^* (E\setminus G) + \lambda^* (([a, b] \setminus E) \setminus G) \geq \lambda^* ([a, b] \setminus G). \qquad (1)$$

Since we know from Royden and Fitzpatrick that G is measurable (in the sense of Royden and Fitzpatrick) and so G satisfies the splitting condition and

$$\lambda^* (E) = \lambda^* (E \cap G) + \lambda(E \setminus G)) \qquad (2)$$

(the splitting condition on G applied to set E) and

$\lambda^*([a, b] \setminus E) = \lambda^*(([a, b] \setminus E) \cap ([a, b] \setminus G)) + \lambda^*(([a, b] \setminus E) \setminus ([a, b] \setminus G))$ (splitting condition on $[a, b] \setminus G$ applied to set $[a, b] \setminus E$)

$$= \lambda^*(([a, b] \setminus G) \setminus e) + \lambda^*(G \setminus E) \text{ since } G \subset [a, b]. \qquad (3)$$

Since E is measurable, by countable additivity

$\lambda^* ([a, b]) = \lambda^* ([a, b] \cap E) + \lambda^* ([a, b] \setminus E) = \lambda^* (E) + \lambda^* ([a, b] \setminus E).$

Therefore

$$\lambda([a, b]) = \lambda^* ([a, b] - \lambda^* (E) + \lambda^* ([a, b] \setminus E)$$

$$= (\lambda^* (E \cap G) + \lambda^* (E \setminus G) + \lambda^* ([a, b] \setminus E)$$

since from G is measurable, from (2)

$$= \lambda^* (E \cap G) + \lambda^* (E \setminus G) + (\lambda^* (([a, b] \setminus G) \setminus E) + \lambda^* (G \setminus E))$$

since $[a, b] \setminus$ is measurable, from (3)

$= (\lambda^* (E \cap G) + \lambda^* (G \setminus E)) + (\lambda^* (E \setminus G) + \lambda^* (([a, b] \setminus G) \setminus E))$

$\geq \lambda^* (G) + \lambda^* ([a, b] \setminus G)$ by monotonocity, since $G = (E \cap G) \cup (G \setminus E)$ and $(([a, b] \setminus G) \setminus E) \cup (E \setminus G) = [a, b] \setminus G$

$= \lambda^* ([a, b] \cap G) \, {}^{*}_{\lambda} \, ([a, b] \setminus G)$ (since $G \subset [a, b]$)

$= \lambda^* ([a, b]) = \lambda([a, b])$ since G is measurable

—splitting condition on [a, b] applied to setG).

Therefore the inequality reduces to equality and

$\lambda^* (E \cap G) + \lambda^* (G \setminus E) + \lambda^* (E \setminus G) + \lambda^* (([a, b] \setminus G) \setminus E)$

$= \lambda^* (G) + \lambda^* ([a, b] \setminus G).$

Subtracting (1) from both sides yields

$$\lambda^* (E \cap G) + \lambda^* (G \setminus E) \leq \lambda^* (G). \qquad (4)$$

Since $A \subset G$, we have $A \cap E \subset G \cap E$ and $A \setminus E \subset G \setminus E$, and by monotonicity

$\lambda^* (A \cap E) + \lambda^* (A \setminus E) \leq \lambda^* (G \cap E) + \lambda^* (G \setminus E)$

$\leq \lambda^* (G)$ (by (4))

$= \lambda^* (A)$ (since G is a measurable content of A).

Combining this with our first inequality, we have established

$$\lambda^* (A) = \lambda^* (A \cap E) + \lambda^* (A \setminus E)$$

for all $A \subset [a, b]$. Therefore the splitting condition is satisfied on E applied to arbitrary set $A \subset [a, b]$.

Note: Nowhere in the previous proof did we use the fact that [a, b] is the smallest interval containing set E. We can therefore state:

Corollary 1: If $E \subset R$ is a bounded measurable set (i.e., $\lambda_* (E) = \lambda^* (E)$), then for any bounded set A we have

$$\lambda^* (A) = \lambda^* (A \cap E) + \lambda^* (A \setminus E).$$

Note: Since we (following Bruckner, Bruckner, Thomson) have defined unbounded set E to be measurable if, for any finite interval I, set $E \cap I$ is measurable, we can extend the previous corollary by eliminating the boundedness restriction:

Corollary 2: If $E \subset \mathbb{R}$ is a measurable set (i.e., $\lambda_*(E) = \lambda^*(E)$), then for any set $A \subset \mathbb{R}$ we have

$$\lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \setminus E).$$

Note: In conclusion, we have shown that a set $E \subset \mathbb{R}$ is measurable (i.e., $\lambda_*(E) = \lambda^*(E)$) if and only if the Carathéodory splitting condition is satisfied:

$\lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \setminus E)$.

Therefore the inner/outer measure definition of Lebesgue measurability (Bruckner/Bruckner/Thomson's) is equivalent to the splitting condition approach.

7.3 LET SUM UP

In this unit we have studied Lebesgue Outer Measure Theorem in detail with examples.we have also studied Lebesgue Measurebility and mnotonically lebesgue outer measures with examples.

7.4 KEYWORD

Approach

Determinancy

Vitali

Axiom

Borel's set

Spprema

Infimum

7.5 QUESTIONS FOR REVIEW

1. Any open or closed interval $[a, b]$ of real numbers is Lebesgue-measurable, and its Lebesgue measure is the length $b - a$. The open interval (a, b) has the same measure, since the difference between the two sets consists only of the end points a and b and has measure zero.

2. Any Cartesian product of intervals [a, b] and [c, d] is Lebesgue-measurable, and its Lebesgue measure is $(b - a)(d - c)$, the area of the corresponding rectangle.

3. Moreover, every Borel set is Lebesgue-measurable. However, there are Lebesgue-measurable sets which are not Borel sets.

4. Any countable set of real numbers has Lebesgue measure 0.

5. In particular, the Lebesgue measure of the set of rational numbers is 0, although the set is dense in **R**.

6. The Cantor set is an example of an uncountable set that has Lebesgue measure zero.

7. If the axiom of determinacy holds then all sets of reals are Lebesgue-measurable. Determinacy is however not compatible with the axiom of choice.

8. Vitali sets are examples of sets that are not measurable with respect to the Lebesgue measure. Their existence relies on the axiom of choice.

9. Osgood curves are simple plane curves with positive Lebesgue measure (it can be obtained by small variation of the Peano curve construction). The dragon curve is another unusual example.

10. Any line in , for , has a zero Lebesgue measure. In general, every proper hyperplane has a zero Lebesgue measure in its ambient space.

7.6 SUGGESTED READINGS & REFERENCES

1. The term volume is also used, more strictly, as a synonym of 3-dimensional volume
2. ^ Henri Lebesgue (1902). "Intégrale, longueur, aire". Université de Paris.
3. ^ Royden, H. L. (1988). Real Analysis (3rd ed.). New York: Macmillan. p. 56. ISBN 0-02-404151-3.
4. ^ Asaf Karagila. "What sets are Lebesgue-measurable?". math stack exchange. Retrieved 26 September 2015.

5. ^ Asaf Karagila. "Is there a sigma-algebra on R strictly between the Borel and Lebesgue algebras?". math stack exchange. Retrieved 26 September 2015.
6. ^ Osgood, William F. (January 1903). "A Jordan Curve of Positive Area". Transactions of the American Mathematical Society. American Mathematical Society. **4** (1): 107–112. doi:10.2307/1986455. ISSN 0002-9947. JSTOR 1986455.
7. ^ Carothers, N. L. (2000). Real Analysis. Cambridge: Cambridge University Press. p. 293. ISBN 9780521497565.
8. ^ Solovay, Robert M. (1970). "A model of set-theory in which every set of reals is Lebesgue-measurable". Annals of Mathematics. Second Series. **92** (1): 1–56. doi:10.2307/1970696. JSTOR 1970696
9. Aliprantis, C.D.; Border, K.C. (2006). Infinite Dimensional Analysis (3rd ed.). Berlin, Heidelberg, New York: Springer Verlag. ISBN 3-540-29586-0.
10. Carathéodory, C. (1968) [1918]. Vorlesungen über reelle Funktionen (in German) (3rd ed.). Chelsea Publishing. ISBN 978-0828400381.

7.7 ANSWERS TO CHECK YOUR PROGRESS

1. Please check section 7.3